Mathematical Induction

MAT230

Discrete Mathematics

Fall 2019

Outline

Motivation

Suppose we were presented with the formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

but were not shown how it was derived. How could we prove that it holds for all integers $n \geq 1$? We could try a bunch of different values of n and find that it works, but how can we know it works for *all* $n \geq 1$?

Mathematical Induction

This sort of problem is solved using **mathematical induction**. Some key points:

- Mathematical induction is used to prove that each statement in a list of statements is true.
- Often this list is countably infinite (i.e. indexed by the natural numbers).
- It consists of four parts:
    - a **base step**,
    - an explicit statement of the **inductive hypothesis**,
    - an **inductive step**, and
    - a **summary statement**.

Mathematical Induction

Here is a list of statements corresponding to the sum we are interested in.

$P(1)$: $1 = (1)(1+1)/2$

$P(2)$: $1 + 2 = (2)(2+1)/2$

$P(3)$: $1 + 2 + 3 = (3)(3+1)/2$

$\vdots$

$P(n-1)$: $1 + 2 + 3 + \cdots + (n-1) = (n-1)((n-1)+1)/2$

$P(n)$: $1 + 2 + 3 + \cdots + (n-1) + n = (n)(n+1)/2$

Notice that the list is *finite* as it contains n statements. However, proving all these are true for any positive integer n means that we have proved an infinite number of statements.

Mathematical Induction

Using mathematical induction is a bit like setting up cascading dominos:

- We begin by proving $P(1)$ is true. This is the **base step**.
- We next show that $P(k) \to P(k+1)$ for $k \geq 1$. This is the **inductive step**.
- We now know that $P(1) \to P(2)$. But as soon as $P(2)$ is known to be true we can say $P(2) \to P(3)$. But then

$$\begin{aligned} P(3) &\to P(4), \\ P(4) &\to P(5), \\ P(5) &\to P(6), \\ &\vdots \end{aligned}$$

Mathematical Induction

In general, the *Principle of Mathematical Induction*, or PMI, is used to prove statements of the form

$$\forall n \geq a, P(n)$$

or, in words, "for all $n \geq a$, the propositional function $P(n)$ is true."

Then

- the base step consists of proving $P(a)$ is true
- the inductive step consists of proving that $P(k) \to P(k+1)$ for any $k \geq a$.

Mathematical Induction Proof

Proposition

$1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ *for any* $n \in \mathbb{Z}^+$.

Proof.

We prove this by mathematical induction.

(**Base Case**) When $n = 1$ we find

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

so the statement is true when $n = 1$.

Mathematical Induction Proof

Proof (continued).

We now want to show that if

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

for some $k \in \mathbb{Z}^+$ then

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

(Note: Stating this does not *prove* anything, but is very helpful. It forces us to explicitly write what we are going to prove, so both we (as prover) and the reader know what to look for in the remainder of the inductive step.)

Mathematical Induction Proof

Proof (continued).

(**Inductive Hypothesis**) Suppose $1 + 2 + \cdots + k = \dfrac{k(k+1)}{2}$ for some $k \in \mathbb{Z}^+$.

(**Inductive Step**) Then

$$\begin{aligned}
1 + 2 + \cdots + k &= \frac{k(k+1)}{2} \\
1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Mathematical Induction Proof

Proof (continued).

(**Summary**) It follows by induction that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ for all $n \in \mathbb{Z}^+$. $\square$

Mathematical Induction Logic

Notice that mathematical induction is an application of Modus Ponens:

$$(P(1)) \wedge (\forall k \in \mathbb{Z}^+, (P(k) \rightarrow P(k+1))) \rightarrow (\forall n \in \mathbb{Z}^+, P(n))$$

Some notes:

- The actual indexing scheme used is unimportant. For example, we could start with $P(0)$, $P(2)$, or even $P(-1)$ rather than $P(1)$. The key is that we start with a specific statement, and then prove that any one statements implies the next.
- To prove something about statements indexed by the integers, the inductive step would include two parts

$$P(k) \rightarrow P(k+1) \quad \text{and} \quad P(k) \rightarrow P(k-1).$$

Example 2

Recall that $a|b$ means "a divides b." This is a proposition; it is true if there is a nonzero integer k such that $b = ka$ otherwise it is false.

Proposition

Show that $3|(n^3 - n)$ *whenever* n *is a positive integer.*

Proof.

We use mathematical induction. When $n = 1$ we find $n^3 - n = 1 - 1 = 0$ and $3|0$ so the statement is proved for $n = 1$.

Now we need to show that if $3|(k^3 - k)$ for some integer $k > 0$ then $3|((k+1)^3 - (k+1))$.

Example 2

Proof (continued).

Suppose that $3|(k^3 - k)$. Then $(k^3 - k) = 3a$ for some integer a. Then, starting with $(k+1)^3 - (k+1)$, we find

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 3a + 3k^2 + 3k \\ &= 3(a + k^2 + k).\end{aligned}$$

We know that $a + k^2 + k$ is an integer so we see that $(k+1)^3 - (k+1)$ is a multiple of 3. By definition, therefore, $3|((k+1)^3 - (k+1))$.

It follows by induction that $3|(n^3 - n)$ for all integers $n > 0$. □

Strong Mathematical Induction

Sometimes it is helpful to use a slightly different inductive step. In particular, it may be difficult or impossible to show $P(k) \to P(k+1)$ but easier (or possible) to show that

$$(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \to P(k+1).$$

In other words, assume that all statements from $P(1)$ to $P(k)$ have been proved and use some combination of them to prove $P(k+1)$.

This is called **strong mathematical induction**.

Strong Mathematical Induction Example

Proposition

Any integer $n > 11$ can be written in the form $n = 4a + 5b$ for $a, b \in \mathbb{Z}$.

Proof.

We use mathematical induction. Let $P(n)$ be the statement "n can be written in the form $4a + 5b$ for some $a, b \in \mathbb{Z}$" and note that

$$
\begin{aligned}
P(12) &: 12 = 4(3) + 5(0) \\
P(13) &: 13 = 4(2) + 5(1) \\
P(14) &: 14 = 4(1) + 5(2) \\
P(15) &: 15 = 4(0) + 5(3)
\end{aligned}
$$

to see that $P(12), P(13), P(14)$, and $P(15)$ are all true.

Strong Mathematical Induction Example

Proof (continued).

Now, suppose that $P(k-3), P(k-2), P(k-1)$, and $P(k)$ have all been proved. This means that $P(k-3)$ is true, so we know that $k-3 = 4a+5b$ for some integers a and b. Adding 4 to both sides we have

$$\begin{aligned} k-3+4 &= 4a+5b+4 \\ k+1 &= 4(a+1)+5b \end{aligned}$$

so $P(k+1)$ must be true, completing the induction. □

Another Mathematical Induction Example

Proposition

$9|(10^n - 1)$ *for all integers* $n \geq 0$.

Proof.

(By induction on n.) When $n = 0$ we find $10^n - 1 = 10^0 - 1 = 0$ and since $9|0$ we see the statement holds for $n = 0$. Now suppose the statement holds for all values of n up to some integer k; we need to show it holds for $k + 1$. Since $9|(10^k - 1)$ we know that $10^k - 1 = 9x$ for some $x \in \mathbb{Z}$. Multiplying both sides by 10 gives

$$\begin{aligned} 10 \cdot (10^k - 1) &= 10 \cdot 9x \\ 10^{k+1} - 10 &= 9 \cdot 10x \\ 10^{k+1} - 1 &= 9 \cdot 10x + 9 \\ &= 9 \cdot (10x + 1) \end{aligned}$$

which clearly shows that $9|(10^{k+1} - 1)$. This completes the induction. □

Fibonacci Numbers

The *Fibonacci sequence* is usually defined as the sequence starting with $f_0 = 0$ and $f_1 = 1$, and then recursively as $f_n = f_{n-1} + f_{n-2}$.

The Fibonacci sequence starts off with 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Fibonacci Numbers

Proposition

Prove that $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ *for* $n \geq 2$.

Proof.

We use induction. As our base case, notice that $f_0 + f_1 = f_3 - 1$ since

$$f_0 + f_1 = 0 + 1 = 1, \qquad \text{and} \qquad f_3 - 1 = 2 - 1 = 1.$$

Suppose that $f_0 + f_1 + f_2 + \cdots + f_k = f_{k+2} - 1$ for some $k \geq 2$. Adding f_{k+1} on both sides, we have

$$\begin{aligned} f_0 + f_1 + f_2 + \cdots + f_k + f_{k+1} &= f_{k+2} - 1 + f_{k+1} \\ &= f_{k+1} + f_{k+2} - 1 \\ &= f_{k+3} - 1. \end{aligned}$$

This completes the induction. □