Inline Problem Solutions

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Inline Solutions to

An Active Introduction to Discrete Mathematics and Algorithms

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1 Motivation

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2 Proof Methods

2.1 Direct Proofs

2.4. $2d + 1$; $c + d + 1$; even.

2.6. $2n$; $2o + 1$; some integers n and o; $2(2no + n)$; $2no + n$; even.

2.7. *Proof.* Let a and b be even integers. Then $a = 2m$ and $b = 2n$ for some integers m and n. Given that, we see that $a \cdot b = (2m)(2n) = 2(2mn)$. Since $2mn$ is an integer, $a \cdot b$ is even. □

2.8. The product $(2n+1)(2q+1)$ should be $4nq+2n+2q+1$ which reduces to $2(2nq+n+q)+1$. In this case the conclusion of parity happens to be correct, but the error in calculation invalidates the proof.

2.9. The problem lies in the second sentence: factoring 2 out of $(2n + 2m + 1)$ does not result in the product of 2 and an integer; the expression $(n + m + 1/2)$ is a *rational* number, not an integer.

2.13. an integer; $3x + 2$; $5x - 7$; 7; 7 divides $15x^2 - 11x - 14$.

2.15. This proof is correct as written.

2.17. Until the conclusion, the reasoning is correct. However, if $k = 1$ then we have not shown that 2 is composite, so we have not shown that all positive even integers are composite.

2.18. *Proof.* By the definition of prime numbers, we know that 2 is prime, since its only factors are 1 and 2. Suppose that $a > 2$ is an even integer, then $a = 2k$ for some integer $k > 1$. This means that a has factors of 2 and k, neither of which is 1, so a is composite. Thus, the only even prime number is 2. □

2.19. We don't need to consider 0 or negative even numbers since by definition only positive numbers can be prime.

2.23. While it is true that $a|(n-1)!$ and $b|(n-1)!$, we cannot conclude from this that $ab|(n-1)!$. For example, $2|12$ and $4|12$ but $8 \nmid 12$. Also, the argument provided here does not work if n is a perfect square.

2.25. When $n > 4$ is composite we can write $n = a \cdot b$ for some positive integers a and b, both greater than 1 and less than $n - 1$. If n is not a perfect square, then $a \neq b$, and we can take a to be the smaller of the two numbers.

2.26. If n is a perfect square then there is a positive integer a such that $n = a^2$. The smallest perfect square greater than 4 is 9, so $a \geq 3$. Therefore, it is correct to say $a > 2$.

2.28. (1) experiment, (2) read (and study) example proofs, (3) practice, Practice, PRACTICE!

2.2 Implication and Its Friends

2.33. The implication "If you read *xkcd*, then you will laugh" is false when (1) you do read *xkcd* AND (2) you do not laugh.

2.34. The implication "If you build it, they will come" can only be false if (1) you do build it AND (2) they do not come.

2.38. "If you do not know a programming language, then you do not know Java."

2.40. true; $\neg p$; false; p; q is false; p is true.

2.42. "If you do not know Java, then you do not know a programming language."

2.43. No, a proposition and its inverse are not equivalent. Suppose you know Python but not Java. The implication "If you know Java, then you know a programming language" is true (because the assumption it makes does not apply), but the inverse is false (because you do know a programming language).

2.45. "If you know a programming language, then you know Java."

2.46. No, a proposition and its converse are not equivalent. Again, suppose you know Python but not Java. The original implication is true but the converse is false.

2.48. (a) Hopefully there are many things that will make you happy that do not have anything to do with watching "The Army of Darkness." So, while watching it may make you happy, your happiness does not depend on watching it.

(b) The original proposition says happiness follows from watching "The Army of Darkness." We can reason, therefore, that if one isn't happy, they didn't get to watch "The Army of Darkness." Likewise, the contrapositive says that unhappiness results from not being able to watch "The Army of Darkness," so if someone did get to watch "The Army of Darkness," they will be happy.

2.3 Proof by Contradiction

2.51. $\sqrt{35}$; $10\sqrt{35}$; $3481 \geq 3500$; false

2.52. *Proof 1:* To use contradiction here we should assume that the conclusion is false, i.e., that both a and b are odd, and show a contradiction arises.

Proof 2: This proves the converse of the proposition since it assumes the conclusion is true and shows that the premise (the original assumption, often called a hypothesis) is true.

Proof 3: This is a correct proof by contradiction of the proposition.

2.56. $1, 2, 3,\ 1, 3, 2,\ 2, 1, 3,\ 2, 3, 1,\ 3, 1, 2,\ 3, 2, 1$

2.58. The proof began by assuming the product was odd because that is the negation of the conclusion. If we can show that this leads to a contradiction, then we know that it must not be the case, and therefore the product must be even.

2.59. We know that $S = 0$ because addition is associative and commutative so $S = (a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = a_1 + a_2 + \cdots a_n - 1 - 2 - \cdots - n$. Since the numbers a_1 through a_n are a permutation of $1, 2, \ldots, n$, we see that $S = 0$.

2.62. Let n be the number of prime factors of a. Each of these prime factors appears exactly twice in a^2, so we know that the number of prime factors of a^2 is even. The same arguments holds for b^2.

2.65. 1. no; 2. yes; 3. no; 4. no; 5. "p implies q" is only false when p is true but q is false; 6. no; 7. No, "p implies q" is never false if p is false. This is because the statement does not apply – since it must be either true or false, and it is not false, it must be true.

2.66. $a > b$; $\frac{a-b}{2}$; $\frac{a-b}{2}$; $\frac{a+b}{2}$; multiply both sides by 2 and subtract a from both sides; $a > b$; contradiction; $a \leq b$

2.68. $a\frac{p^2}{q^2} + b\frac{p}{q} + c$; multiply both sides by q^2; odd; zero; the terms with p are even and the other term is odd and the sum of two even integers and an odd integer is odd, which contradicts the fact that the sum is zero; the terms with q are even and the other term is odd and the sum of two even integers and an odd integer is odd, which contradicts the fact that the sum is zero; we know that there is no rational number solution to $ax^2 + bx + c = 0$.

2.4 Proof by Contraposition

2.72. The proposition we are trying to prove is "If n is an integer and $3n + 2$ is even, then n is even" so the contrapositive is "If n is an odd integer, then $3n + 2$ is odd."

Proof 1: Rather than working with the contrapositive, this is a proof of the converse: "if n is an even integer, then $3n + 2$ is even.

Proof 2: This attempts to proof the correct statement and goes well until the last step. The problem is that $\frac{6}{5}k + 1$ is not an integer, so we cannot invoke the definition of odd numbers.

Proof 3: Like Proof 2, this starts off well, and the only real problems is that technically we cannot say $6k + 5$ is odd using the definition of odd. What should be done is to say $6k + 5 = 6k + 4 + 1 = 2(3k + 2) + 1$ which is odd. This correctly shows that $6k + 5$ can be written in the form $2m + 1$ for an integer m.

2.5 Other Proof Techniques

2.78. Both 3 and 5 are prime numbers, but the sum $3 + 5 = 8$ is not prime. Therefore, it is not true that the sum of any two primes is also prime.

2.81. the proof is complete since $s \in [s, 2s]$; if we multiply this inequality by 2 we have $2^r < 2s < 2^{r+1}$ so we see $s < 2^r < 2s$. Thus, for any positive integer s, there is a power of 2 in $[s, 2s]$.

2.6 If and Only If Proofs

2.82. Since $q \to p$ is equivalent to the contrapositive statement $\neg p \to \neg q$, we can prove either of these to establish the second part of an iff proof.

2.83. *Proof.* Let x be an odd integer so that $x = 2k + 1$ for some integer k. Then $x + 20 = 2k + 21 = 2(k + 10) + 1$, which is odd since $k + 10$ is an integer. Conversely, if $x + 20$ is odd then $x + 20 = 2k + 1$ for some integer k. In this case $x = 2k - 19 = 2(k - 10) + 1$, and since $k - 10$ is an integer we see that x must be odd. Therefore x is odd iff $x + 20$ is odd. □

2.85. *Proof.* Let x be an odd integer so that $x = 2k + 1$ for some integer k. Then $x + 20 = 2k + 21 = 2(k + 10) + 1$, which is odd since $k + 10$ is an integer. Conversely, if x is even then $x = 2k$ for some integer k so $x + 20 = 2k + 20 = 2(k + 10)$. This is even since $k + 10$ is an integer. Therefore x is odd iff $x + 20$ is odd. □

2.86. p implies q; q implies p; p implies q; $\neg p$ implies $\neg q$

2.7 Common Errors in Proofs

2.89. The writer represents two integers that may be different with the variable a. The problem is that, as written, the proof requires that x and y must be equal. The logic is otherwise correct, changing the representation of y to be $y = 2b$ for some integer b and following this through would yield a correct proof.

2.90. Here again the writer represents two integers that may be different with the variable a. Unlike the last example this mistake allows the writer to "prove" something that is not true. If the representation of y is changed to $y = 2b$ for some integer b we find that $x + y = 2(a + b)$ which is divisible by 2 but not necessarily by 4.

2.91. The proof is incorrect because the result is clearly false.

2.92. In the final statement of a proof you should not work both sides of an equation. Note: It is often helpful to do this while figuring out how to write the proof, but this work should then be rewritten so that only one side of the equation is manipulated or operations that equivalently change both sides (e.g. multiplying both sides by 2) are carried out. If we are able to correctly do this, we can be sure that equation is true.

2.93. The transitive property says that if $a = b$ and $b = c$ then $a = c$, so I think a case could be made that it is okay to do this. However, the author is making the point that subtle errors can creep in (e.g. saying $x^2 = 1$ implies only $x = 1$ when it could also imply $x = -1$). Again, I think that it is okay to work with both sides while exploring how to write the proof, but the final version should avoid doing that.

2.8 More Practice

2.94. *Proof.* Let $p < q$ be two consecutive odd primes (this means neither p nor q is 2). Regardless of their being prime numbers, both p and q are odd so there are two positive integers m and n such that $p = 2m + 1$ and $q = 2n + 1$. Adding we find $p + q = (2m + 1) + (2n + 1) = 2(m + n + 1)$. Note that $m + n + 1$ is a positive integer greater than 1, so $p + q$ is composite.

Not only is $p + q$ composite, we have just shown that it is even. Thus $(p + q)/2$ is an an integer. Since this integer is the average of p and q it must be the case that $p < (p + q)/2 < q$. As p and q are consecutive primes, we know that $(p + 2)/2$ is composite, and so has at least two prime factors, which we call l and k. This means $p + q = 2lk$, so $p + q$ has at least three, not necessarily distinct, prime factors. □

2.95. *Proof 1:* There is an error in the final statement that makes this proof circular. For a^y/b^y to be rational we'd need to know that both a^y and b^y are integers, or least rational themselves. However, this is exactly what we're trying to prove.

Proof 2: To say "x^y is just x multiplied by itself y times" is not correct. If y is a non-integer rational then this statement does not make since. Actually, if you think about it, it is ambiguous at best even when y is an integer.

2.96. *Disproof.* Let $x = 2$ and $y = 1/2$. Then $x^y = 2^{1/2} = \sqrt{2}$, which is irrational. Thus, x^y may not be rational. □

2.97. *Proof 1:* This is gibberish from beginning to end.

Proof 2: Describing a rational as "an integer over an integer" demonstrates a lack of understanding as to what a rational number is – and the use of "over" to mean "divided by" can be confusing. The conclusion sentence makes a huge leap that, even if we make allowances for the the language, would need to be proved as it is essentially what we're trying to prove in the first place.

Proof 3: This is better. However, the contrapositive of the original statement would be "if $1/x$ is rational, then x is irrational." The prover almost did this, but should have started with "Since it is rational, $1/x = p/q$..." Then, of course, there is the problem about x being zero...

Proof 4: This is even better and is just about correct. The only real problem is that the form b/a requires that $a \neq 0$. Yes, it is true that $1/x$ is not zero for any integer x so we understand that a/b is not zero, and so $a \neq 0$, but this is a lot of reasoning to leave to the reader in the midst of a proof, and it should be explicitly pointed out.

2.98. *Proof 1:* This proof attempt starts of by restating the proposition, but does so incorrectly. The proposition are asked to consider is "if p is a prime number, then $2^p - 1$ is prime," but the writer states the converse and tries a proof by contradiction. If $p = st$ is not prime, then we need to require that we can find both s and t to be integers greater than 1 and less than p. The formula for expanding $2^{st} - 1$ is correct.

Proof 2: The first two statements made, assuming that p is prime (or at least an integer). However, it is certainly not the case that odd numbers do not have any factors.

2.99. *Proof.* Consider prime factorizations of $2^p - 1$ when p is prime:

p	2	3	5	7	11
prime factorization of $2^p - 1$	3	7	31	127	$23 \cdot 89$

Thus $2^{11} - 1 = 2047$ is not prime, so not all numbers of the form $2^p - 1$, where p is prime, are prime. □

3 Programming Fundamentals and Algorithms

3.1 Algorithms

3.2. The body of the function is a single line:

```
double areaSquare(double w) {
  return w * w;
}
```

3.7. The algorithm does not work correctly. The first line assigns the original value of `y` to `x`, overwriting the value originally in `x`. The second line then assigns the current (new) value of `x` to `y`; these values are the same. Upon termination both variables contain the value originally stored in `y`.

3.12. (a) 45; (b) 8; (c) 3; (d) 6; (e) 0; (f) 7; (g) 7; (h) 7; (i) 11

3.21. −15; −7; 9; 13; 21

3.22. The compiler could return -1 or 3.

3.23. *Solution 1:* The offered solution focuses on the range of the modulo function. It ensures that the result is in the correct range, but will usually return an incorrect value. positive values from $a \bmod b$ should be unchanged, but are halved and then shifted by $(b-1)/2$. Similarly, negative values should be shifted by b but instead they are halved and shifted by $(b-1)/2$.

Solution 2: This works correctly if $a \bmod b$ already returns a nonnegative value, but does not handle the other case correctly.

Solution 3: This works correctly but uses a conditional, which we were instructed to avoid if possible.

Solution 4: This will always return the same value as $a \bmod b$. In particular, it does not map a negative result to the correct positive result.

3.24. A correct version of the algorithm is $((a \bmod b)+b) \bmod b$. Note that while this avoids a conditional, it is more expensive in that it requires two integer modulo operations. A "less than zero" conditional check is probably more computationally efficient even if it makes the code seem slightly longer. There are instances, however, where avoiding conditionals is rather important.

3.27. 1. 9; 2. 10; 3. 9; 4. 10; 5. 9; 6. 9

3.29. *Solution 1:* This incorrectly rounds numbers like 0.501, which should round up to 1 but this will round down to 0. The mistake is assuming that 0.49 is the smallest number less than 0.5 that the computer can represent.

Solution 2: This merely truncates positive real numbers since the literal 1/2 will be evaluated using integer division, yielding 0.

Solution 3: This incorrectly rounds numbers like 0.2 up to 1.

Solution 4: This works correctly when $x \geq 0$. Consider the interval $[0, 0.5]$. Subtracting 0.5 results in a number from $[-0.5, 0]$ and the ceiling function on this returns 0. Next, consider numbers from $(0.5, 1)$. The subtraction maps these to $(0, 0.5)$, and the ceiling function on this range will return 1.

3.32. *Solution 1:* This will always return the same truncated value as n/m does and so does not round correctly. It also requires the use of floating point arithmetic which is undesirable.

Solution 2: Since `1/2` will evaluate to 0, this also returns the same truncated value as n/m does, but at least now no floating point is required.

Solution 3: This does the same thing as Solution 1.

3.33. One possible integer-only solution is `(2*n/m+1)/2`.

3.36.

```
int max(int x, int y, int z) {
    return max(max(x,y),z);
}
```

This algorithm can be broken down into two steps. The first, computed with `max(x,y)`, returns the larger of x and y. All that remains is to compare this value with z and return the larger of the two; this is what the second call to `max()` does.

3.37.

```
void HelloGoodbye(int x) {
  if ( x >= 4 ) {
    if ( x <= 6 ) {
      print("Hello");
    } else {
      print("Goodbye");
    }
  } else {
      print("Goodbye");
  }
}
```

3.38.

```
void HelloGoodbye(int x) {
  if ( x >= 4 ) {
    if ( x <= 6 ) {
      print("Hello");
      return;
    }
  }
  print("Goodbye");
}
```

3.41. The factorial function presented works correctly when $n \geq 0$. When $n < 0$ the function will return 1. To see why, note that since $n \neq 0$ control passes to the `else` block where `fact` is initialized to 1. Since $i = 1$, which is greater than any negative number, the for-loop condition is false so the loop body is not executed and the current value of `fact` is returned. Since the factorial function is not defined for negative values we could check if $n < 0$ and signal an error.

3.42. (a) *Solution 1:* Since $i = 0$ the first time through the loop body the value of `fact` becomes 0. Once zero, it remains equal to zero so this function always returns 0.

(b) *Solution 2:* This works correctly for $n \geq 0$

(c) *Solution 3:* This works correctly for $n \geq 0$

(d) *Solution 4:* This actually computes $(n-1)!$ when $n > 1$. It would be correct if the loop body was `fact = fact*(n-i+1)`.

3.43.

```
double power(double x, int n) {
  double prod = 1.0;
  for (int i=0; i<n; i++) {
    prod *= x;    // same as prod = prod * x;
  }
  return prod;
}
```

3.48. Since n is an integer and 2 is a literal integer, the computation $(n-2)/2$ is performed using integer-only arithmetic. In C/C++ the division will result in a truncated integer value, which is the same value that the floor function returns (at least for positive integers).

3.49. Yes, this algorithm works correctly. The for-loop condition is `i<n/2`, which is equivalent to the condition `i<=(n-2)/2` used in Example 3.47.

3.51. Let's generate a table of data:

n	-3	-2	-1	0	1	2	3	4
$\lceil n/2 \rceil - 1$	-2	-2	-1	-1	0	0	1	1
$\lfloor (n-1)/2 \rfloor$	-2	-2	-1	-1	0	0	1	1

Based on this table it does appear that these two expressions are equivalent. A proof might look like the following:

Proof. We consider two cases, n even and n odd. To start, suppose n is even so $n = 2k$ for some integer k. Then

$$\begin{aligned} \lceil n/2 \rceil - 1 &= \lceil 2k/2 \rceil - 1 &&= k - 1 \\ \lfloor (n-1)/2 \rfloor &= \lfloor (2k-1)/2 \rfloor = \lfloor k - 1/2 \rfloor &&= k - 1 \end{aligned}$$

Now suppose n is odd so $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} \lceil n/2 \rceil - 1 &= \lceil (2k+1)/2 \rceil - 1 = \lceil k + 1/2 \rceil - 1 = k + 1 - 1 &&= k \\ \lfloor (n-1)/2 \rfloor &= \lfloor (2k)/2 \rfloor &&= k \end{aligned}$$

Thus $\lceil n/2 \rceil - 1 = \lfloor (n-1)/2 \rfloor$ for all integers n. □

3.57. The following table shows the results of tracing the algorithm with $n = 5$:

n	5	3	1
x	0	10	16
i	1	2	3
$n \cdot i$	5	6	3
$n \cdot i > 4$	T	T	F
$n > 1$	T	T	F

The value returned is 16.

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4 Logic

4.1 Propositional Logic

4.3. (a) false; (b) true; (c) true; (d) false.

4.5. (a) not a proposition; (b) could be true or false depending on the person, so this is not a proposition; (c) not a proposition; (d) true; (e) false.

4.6. (a) *Proof.* The statement "This is a proposition" is a true statement. It must be, therefore, that it's negation is also a proposition, but one that is false. The negation is "This is not a proposition," so we see that this is indeed a proposition but that it is false. □

(b) *Proof.* Assume that "This is not a proposition" is not a proposition. By definition, however, This means that the statement is a proposition, contradicting the assumption that is not. Since the statement it makes is false, its truth value is false. □

4.10. I am not learning discrete mathematics; false (we hope).

4.11. 1. `list.size() > 0`; 2. `list.size() != 0`

4.15. I like cake and I like ice cream.

4.19. $x > 0$ or $x < 10$; true; true; $x < 10$; true

4.20. 1. Formally the negation of p might read "It is not the case that you must be at least 48 inches to ride the roller coaster." This basically says there is no minimum height. 2. you must be at least 48 inches tall or at least 18 years old to ride the roller coaster; 3. you must be at least 48 inches tall and at least 18 years old in order to ride the roller coaster.

4.21.

```
boolean startsOrEndsWithZero(int[] a, int n) {
  return (n > 0 && ( a[0] == 0 || a[n-1] == 0));
}
```

4.22. This solution does work for array sizes $n = 0$ and $n = 1$. In the first case if $n = 0$ then the very first test will be false so the conjunction will be false and the second test (which accesses array elements) is not executed. In the second case the first test of the conjunction is true so the second one is evaluated. Now $n - 1 = 0$ so the element `a[0]` is accessed twice and the returned value if this is zero otherwise the return value is false.

4.25. XOR; OR (could be XOR); OR; OR; XOR

4.26. $p \vee q$ is true if either list is empty and also if both lists are empty. $p \oplus q$ is true only when one of the two lists, but not both, is empty.

4.27. (a) Well, technically no, these are not the same. The first means $x < 5$ or $x > 15$ or both, while the second means $x < 5$ or $x > 15$ but not both. (b) Yes, $p \vee q$ and $p \oplus q$ practically accomplish the same thing here, because if $x < 5$ is true then $x > 15$ will be false, and vice versa.

4.30. (a) You will get an A. If $p \to q$ is true, then if p is true we can conclude that q is also true. (b) We don't know if you got 90%. The implication $p \to q$ is true whenever q is true, so we don't know the truth value of the hypothesis. (c) You may or may not get an A. If the premise is false the proposition will be true regardless of the truth value of the conclusion.

4.33. (a) You will get an A, since the biconditional includes the statement "if you earn 90%, then you will get an A." (b) You earned 90%, since the biconditional includes "if you get an A, then you earned 90%." (c) You did not get an A. The biconditional includes "if you got an A, then you earned 90%" and the contrapositive of this is "if you did not earn 90%, then you do not get an A."

4.35. If *Iron Man* is on TV, then I will watch *Iron Man*; $\neg r \wedge p \to q$; *Iron Man* is on TV and I don't own it on DVD, and I won't watch it; $p \leftrightarrow q$; $r \to q$

4.38.

p	q	$p \to q$	$(p \to q) \wedge q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	F

4.40.

a	b	c	$\neg b$	$a \vee \neg b$	$(a \vee \neg b) \wedge c$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	T	T	T
F	F	F	T	T	F

4.43. $a \wedge b \vee c$ should be interpreted $(a \wedge b) \vee c$

4.44. $(a \wedge b) \vee c$ and $a \wedge (b \vee c)$ are *not* equivalent. Suppose $a = F$, $b = T$, $c = T$. Then $(a \wedge b) \vee c$ is $(F \wedge T) \vee T$ which is true while $a \wedge (b \vee c)$ becomes $F \wedge (T \vee T)$ which is false.

4.45. The writer is correct that implications have left-to-right associativity so that $a \to b \to c$ should be interpreted as $(a \to b) \to c$. They are incorrect, however, in saying this is the same as $a \to (b \to c)$. To see this, consider a case when a and c are both false (the value of b does not matter). Because a is false $a \to (b \to c)$ will be true. But $a \to b$ is also true so that $(a \to b) \to c$ is false.

4.2 Propositional Equivalence

4.48. (a) tautology; (b) contradiction since either p or $\neg p$ will be false; (c) contingency since we don't know the values of p and q

4.50. *Proof 1:* The truth table is correct, but leaves the reader to infer what the meaning of the table is.

Proof 2: This version improves on the last one by explaining to the reader what the truth table is generated for and concludes by saying what the table shows. Some of the wording could be better...

Proof 3: While this proof only handles the case where q is false it does (correctly but a bit confusingly) demonstrate that in this case it is sufficient to only consider the case when q is false. It also makes statements like "...the statement equals out to false implies false,..." which at best are unconventional and imprecise and at worst are meaningless.

Proof 4: This is a good proof.

Proof 5: Really???

4.54. *Proof.* We prove this by showing that the truth table column corresponding to $\neg(p \wedge q)$ is identical to the column for $\neg p \vee \neg q$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since they are same for every row in the table, $\neg(p \wedge q) = \neg p \vee \neg q$. □

4.55. negation; distributive; $p \vee p$; $p \wedge T$; $p \wedge (p \vee \neg p)$; $(p \wedge p) \vee (p \wedge \neg p)$; $(p \wedge p) \vee F$; identity; $p \wedge p = p$

4.57. (a) Prove that $p \vee (p \wedge q) = p$.

Proof.

$$\begin{aligned} p \vee (p \wedge q) &= (p \wedge T) \vee (p \wedge q) && \text{identity} \\ &= p \wedge (T \vee q) && \text{distributive} \\ &= p \wedge T && \text{domination} \\ &= p && \text{identity} \end{aligned}$$

□

(b) Prove that $p \wedge (p \vee q) = p$.

Proof.

$$\begin{aligned} p \wedge (p \vee q) &= (p \vee F) \wedge (p \vee q) && \text{identity} \\ &= (p \vee (F \wedge q) && \text{distributive} \\ &= p \vee F && \text{domination} \\ &= p && \text{identity} \end{aligned}$$

□

4.60.
```
if (x>0) x=y;
```

4.61. The reasoning that we don't need to check `x>0` twice is correct. However, in the original code the condition `x>0` is evaluated before the condition `x<y || x>0` and this condition is true whenever $x > 0$. The proposed simplification allows the assignment `x=y` to happen when $x \leq 0$ provided that $x < y$. This is different behavior than the original code.

4.62. Notice that the the only time the inside if-statement is evaluated is when $x > 0$, so the disjunction in the inner if is always true (absorption). Thus, the code can be simplified to

```
if (x>0) x=y;
```

4.64. *Solution 1:* This solution erroneously assumes the list is not empty and so omits the necessary check for that.

Solution 2: This solution applies DeMorgan's law properly to the first compound proposition but neglects the last condition entirely. As a result the proposed code increments x when $x < 100$ while it would be decremented in the original code.

Solution 3: This is correct.

4.65. The original code actually has a subtle bug since if `list.isEmpty()` is true the value of `list.get(0)` is accessed. Of course, this is an error if the list is empty. Thus, the final solution in the last problem is better than the original code.

4.66. (a) $p \oplus q$

(b) One answer is $(p \vee q) \wedge \neg(p \wedge q)$, another is $(p \wedge \neg q) \vee (\neg p \wedge q)$.

4.68. *Proof 1:* This only deals with one of four possible cases of truth values of p and q.

Proof 2: What does "they" refer to? What about if p and q have different truth values?

Proof 3: This is incorrect as the word "precisely" here implies the propositions $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg q \wedge \neg q)$ are only equivalent when both p and q are true.

Proof 4: Better, but still a bit fuzzy since it refers to the propositions as "these." This "proof" also essentially claims the result. It suggests a good line of reasoning, but leaves the reader to connect the dots.

4.3 Predicates and Quantifiers

The inline problems in this subsection have not been completed.

4.4 Normal Forms

4.108.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

To write $p \leftrightarrow q$ in disjunctive normal form, we use the first and last rows from the truth table to write

$$(p \wedge q) \vee (\neg p \wedge \neg q).$$

4.110. $(\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$

4.5 Bitwise Operations

4.116. 11110000; 11110000; 00001111; 15

4.117. $\sim$11000110 = 00111001 which is 57 in decimal.

4.120. 11000000; 11111100; 00111100

---

5 Sets, Functions, and Relations

5.1 Sets

5.4. $\{2, 3, 5, 7\}$

5.7. $|A| = 6$, $|B| = 5$, $|C| = 6$; A and C represent the same set.

5.10. $|\mathbb{C}| = \infty$, $|\mathbb{Z}^+| = \infty$, $|\varnothing| = 0$

5.13. $\{2k : k \in \mathbb{Z}\}$

5.14. $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$

5.17. (a) $A \subseteq S$ since each element of A is a perfect square

(b) $A \subset S$ since $A \subseteq S$ but S contains elements not in A

(c) $S \subseteq S$ since every element in S is trivially in S

(d) $S \not\subset S$ since there are no elements in S that are not in S.

(e) $S \not\subset A$ since S contains elements that are not in A, such as 25.

5.18. (a) $A \subseteq B$ since every integer divisible by 6 is also divisible by 2.

(b) $A \subseteq C$ since every integer divisible by 6 is also divisible by 3.

(c) $B \not\subseteq A$ since $4 \in B$ but $4 \notin A$.

(d) $B \not\subseteq C$ since $4 \in B$ but $4 \notin C$.

(e) $C \not\subseteq A$ since $3 \in C$ but $3 \notin A$.

(f) $C \not\subseteq B$ since $3 \in C$ but $3 \notin B$.

5.21. There are 16 subsets of $\{a, b, c, d\}$:

$$\begin{array}{cccc} \varnothing & \{a\} & \{b\} & \{c\} \\ \{d\} & \{a,b\} & \{a,c\} & \{a,d\} \\ \{b,c\} & \{b,d\} & \{c,d\} & \{a,b,c\} \\ \{a,b,d\} & \{a,c,d\} & \{b,c,d\} & \{a,b,c,d\} \end{array}$$

5.24. $\{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\},$ $\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$

5.26. (a) $|P(A)| = 2^4 = 16$

(b) $|P(P(A))| = 2^{2^4} = 2^{16} = 65536$

(c) $|P(P(P(A)))| = 2^{2^{2^4}} = 2^{65536}$

5.27. Increasing the cardinality of a finite set A by 1 will **double** the size of the power set. The new power set contains all the subsets of the original set, and also the same number of new subsets identical to the original subsets except each contains the new element added to A.

5.2 Set Operations

5.30. $A \cup B = \mathbb{Z}$

5.33. $A \cap B = \varnothing$

5.36. $A \backslash B = A$; $B \backslash A = B$

5.39. $\overline{A} = B$; $\overline{B} = A$

5.43. The sets A and B are disjoint. Every element of A is even and so does not belong to B. Similarly, every element of B is odd and so is not in A. Alternatively, $A \cap B = A \cap \overline{A} = \varnothing$ so A and B are disjoint.

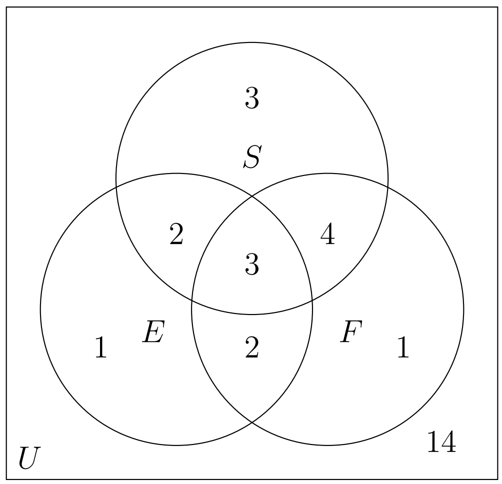
5.47. *Proof 1:* This proof almost starts well but has several problems. Perhaps the most significant is that it attempts to show $A \backslash B \subseteq A \cap \overline{B}$ but neglects to show $A \cap \overline{B} \subseteq A \backslash B$. Another problem is the additional curly braces around $A - B$. This means the proof attempt starts by assuming $x \in \{A - B\}$, which means that x is an element of a set that contains the set $A - B$ as a single element. Thus, $x = A - B$ so saying $x \in A$ is erroneous.

Proof 2: This is very confusing and leaves out key words (e.g. using "universal" rather than "universal set"). We also haven't defined what a "part" of a set is. There are other problems as well...

Proof 3: Well done!

5.49. $x \in (A \cup B) \cap C$
$\leftrightarrow x \in (A \cup B) \wedge \underline{x \in C}$ by the definition of intersection
$\leftrightarrow (x \in A \vee \underline{x \in B}) \wedge x \in C$ by definition of union
$\leftrightarrow (x \in A \wedge x \in C) \vee \underline{(x \in B \wedge x \in C)}$ by the distributive property
$\leftrightarrow \underline{(x \in A \cap C)} \vee (x \in B \cap C)$ by the definition of intersection
$\leftrightarrow x \in (A \cap C) \cup (B \cap C)$. by the definition of union.

5.52. The follow Venn diagram shows that, in this group of 30 people, there are 16 people who speak at least one of the three languages and 14 who speak none of them.



5.55. $A \times B = \{(1,3), (2,3), (3,3), (4,3)\}$

5.58. $A^2 = \{(0,0), (0,1), (1,0), (1,1)\}$
$A^3 = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

5.61. (a) $|A \times B| = 500$
(b) $|A \times C| = 200$
(c) $|A^2| = 100$
(d) $|B^3| = 50^3 = 125000$
(e) $|A \times B \times C| = 10 \cdot 50 \cdot 20 = 10000$

5.3 Functions

5.69. $f : \mathbb{Z} \to \{0,1\}$ where $f(x) = x \bmod 2$

5.73. We can "work on both sides" in Example 5.72 since the algebraic operations performed maintain strict equality between consecutive expressions on each side of the equal sign.

5.74. *Proof.* Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} f(a) &= f(b) \\ 5a &= 5b \\ a &= b \end{aligned}$$

Since asserting $f(a) = f(b)$ requires that $a = b$, we see that f is one-to-one on $\mathbb{R}$. □

5.77. Observe that $f(1.1) = 1$ and $f(1.2) = 1$. Since $f(1.1) = f(1.2)$, the function f is not one-to-one on $\mathbb{R}$.

5.80. *Proof.* Let $b \in \mathbb{R}$. Note that if $b = 2a + 1$, we can solve for a to find that $a = (b-1)/2$. Thus $f(a) = 2((b-1)/2) + 1 = b - 1 + 1 = b$. Since this works for every $b \in \mathbb{R}$, we see that f is onto. □

5.83. Observe that $f(x) = \lfloor x \rfloor$ always returns an integer value, so it is impossible to find any x for which $f(x) = 1.5$. Therefore when the domain is the real numbers, f is not onto. Note that if $f : \mathbb{R} \to \mathbb{Z}$, then this function would be onto.

5.84. (a) The function $f(x) = x^2$ is one-to-one on $\mathbb{Z}$. (False)

Disproof. Note that $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$. Thus, f is not one-to-one on $\mathbb{Z}$. □

(b) The function $f(x) = x^2$ is one-to-one on $\mathbb{R}$. (False)

Disproof. Since $\mathbb{Z} \subseteq \mathbb{R}$, the argument just shown also works here to show that f is not one-to-one on $\mathbb{R}$. □

(c) The function $f(x) = x^2$ is one-to-one on $\mathbb{N}$. (True)

Proof. Let $a, b \in \mathbb{N}$ such that $f(a) = f(b)$. Then $a^2 = b^2$ and so $\sqrt{a^2} = \sqrt{b^2}$. Since $a \geq 0$ and $b \geq 0$ we know that $\sqrt{a^2} = a$ and $\sqrt{b^2} = b$, showing that $a = b$. Therefore f is one-to-one on $\mathbb{N}$. □

5.85. (a) $f(x) = x + 2$ is both one-to-one and onto.

Proof. Suppose $a, b \in \mathbb{Z}$ such that $f(a) = f(b)$. Then $a + 2 = b + 2$ so $a = b$, showing that f is one-to-one. Now let b be any element of $\mathbb{Z}$ and choose $a = b - 2$. Then $f(a) = f(b-2) = b - 2 + 2 = b$, showing that f is onto. □

(b) $g(x) = x^2$ is neither one-to-one nor onto.

Proof. Note that $g(-2) = 4$ and $g(2) = 4$ so g is not one-to-one. Also note that $1 < \sqrt(3) < 2$ so there is no integer a for which $g(a) = 3$, showing that g is not onto. □

(c) $h(x) = 2x$ is one-to-one but not onto.

Proof. Suppose $a, b \in \mathbb{Z}$ such that $h(a) = h(b)$. Then $2a = 2b$ so $a = b$, showing that h is one-to-one. However, we observe if $h(x) = 3$ then $2x = 3$. If x is an integer then the left side is even while we know the right side is odd, clearly a contradiction, so no such integer x exists. Thus, h is not onto. □

(d) $r(x) = \lfloor x/2 \rfloor$ is not one-to-one but is onto.

Proof. Notice that $r(2) = \lfloor 2/2 \rfloor = 1$ and $r(3) = \lfloor 3/2 \rfloor = 1$ so r is not one-to-one. However, if we let b be any integer and choose $a \in \mathbb{Z}$ so that $a = 2b$ then $r(a) = \lfloor 2b/2 \rfloor = b$, we see that r is onto. □

5.87. (a) *False.* We only know that $|A| \geq |B|$, so A and B need not have the same cardinality nor be equal.

(b) *False.* Consider $f(x) = 1$ with $A = \mathbb{Z}$. This is neither one-to-one nor onto.

(c) *True.* Since f is a one-to-one correspondence, every element in A is paired with an element of B and vice-versa. This requires that $|A| = |B|$

(d) *False.* $f(1)$ is multivalued, something not allowed for a function.

(e) *True.* Example 5.79 proved that f is onto. To see that f is one-to-one, let $a, b \in \mathbb{R}$ so that $a^3 = b^3$. Since $\sqrt[3]{a^3}a$ and $\sqrt[3]{b^3} = b$, we see that $a = b$, showing that f is one-to-one.

(f) *False.* Since $f(x)$ is not defined when $x < 0$, it is not true to say the domain of f is $\mathbb{R}$, thus f as defined here is not a valid function. If, however, we restrict the domain to be $\{x : x \in \mathbb{R}, x \geq 0\}$, then f is one-to-one (no two different positive reals have the same square root) but not onto since the square root function does not return any negative values.

(g) *True.* The codomain is the set that contains all the output values of a function while the range *is* the set of output values.

(h) *False.* Many functions that are one-to-one are not onto. One example is shown in (c) of Example 5.85.

(i) *False.* For f to be onto, we would need to be able to find an $x \in \mathbb{Z}$ such that $y = ax + b$ for any $y \in \mathbb{Z}$. Suppose $a = 2$ and $b = 1$ and notice that $y = 2x + 1$ so that y must be odd if x is an integer. Thus, y cannot be an arbitrary integer. This contradiction shows that f is not onto.

(j) *False.* The preceding argument works again here if $\mathbb{Z}$ is replaced by $\mathbb{N}$.

(k) *True (provided $a \neq 0$).* Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$. Then $ax + b = ay + b \to ax = ay \to x = y$ as long as $a \neq 0$ so f is one-to-one. Note as well that if $x = (y - b)/a$ for any $y \in \mathbb{R}$ then $f(x) = y$, showing that f is onto.

5.93. If we set $a = 3$ and $b = -5$ then we can use the work shown in Exercise 5.87 part (k) to prove that f is a one-to-one correspondence and therefore is invertible. Finding the inverse can be done by setting $y = 3x - 5$ and solving for x:

$$\begin{aligned} 3x - 5 &= y \\ 3x &= y + 5 \\ x &= \frac{y+5}{3}. \end{aligned}$$

Thus $f^{-1}(x) = (x + 5)/3$.

5.96. $(f \circ g)(x) = \lfloor x/2 \rfloor$
$(g \circ f)(x) = (\lfloor x \rfloor)/2$

5.101. (a) *False.* In Exercise 5.87 part (i) we showed that $f : \mathbb{Z} \to \mathbb{Z}$ is not onto and so is not a one-to-one correspondence. Therefore f is not invertible.

(b) *False.* Same reasoning again, using Exercise 5.87 part (j) and replacing $\mathbb{Z}$ by $\mathbb{N}$.

(c) *True (provided $a \neq 0$).* Exercise 5.89 part(k) shows that $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one correspondence so f is invertible.

(d) *False.* $1/x^2$ is the reciprocal of x^2, not the inverse.

(e) *False.* This is true if n is an odd positive integer.

(f) *False.* The function $\sqrt[n]{x}$ is not defined on $\mathbb{N}$ since $\mathbb{N}$ cannot be the codomain. For example, $x = 2 \in \mathbb{N}$ but $\sqrt{2} \notin \mathbb{N}$.

(g) *True.* Let $x, y \in \mathbb{R}^+$. Then $\sqrt[n]{x} = \sqrt[n]{y}$ implies $(\sqrt[n]{x})^n = (\sqrt[n]{y})^n$ so $x = y$ and the function is one-to-one. Also, note that for any $y \in \mathbb{R}^+$, if we set $x = y^n$ then $y = \sqrt[n]{x}$ so the function is onto. Thus, $\sqrt[n]{x}$ is invertible on $\mathbb{R}^+$.

(h) *False, as written.* Notice that $g(x) = 1/x$ is not defined on $\mathbb{Z}$ when $x \neq 1$ since the value it yields is not an integer. If, however, we consider functions f and g defined on $\mathbb{Q}$ then it is true that $f \circ g = g \circ f$:

$$(f \circ g)(x) = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2} = (g \circ f)(x).$$

(i) *False.* $f \circ g \neq g \circ f$ since

$$(f \circ g)(x) = ((x+1)+1)^2 = (x+2)^2 = x^2+4x+4,$$
$$(g \circ f)(x) = (x+1)^2+1 = x^2+2x+1+1 = x^2+2x+2$$

and $x^2+4x+4 \neq x^2+2x+2$.

(j) *False.* Note that

$$(f \circ g)(1.5) = \lfloor \lceil 1.5 \rceil \rfloor = \lfloor 2 \rfloor \quad \text{and} \qquad (g \circ f)(1.5) = \lceil \lfloor 1.5 \rfloor \rceil = \lceil 1 \rceil = 1.$$

Since these are different values, we see that $f \circ g \neq g \circ f$.

(k) *False.* Observe that $f(1.5) = 1$ but $g(1) = 1 \neq 1.5$.

(l) *True.* Notice that $(f \circ g)(x) = (\sqrt{x})^2 = x$ and $(g \circ f)(x) = \sqrt{x^2} = x$ so both of these composite functions are the identity function $\iota_{\mathbb{R}^+}$.

5.4 Partitions and Equivalence Relations

5.109. I would use samples from the following sets:

$$\{(x,b) \in \mathbb{Z} \times \mathbb{Z} : x < b\}, \quad \{(x,b) \in \mathbb{Z} \times \mathbb{Z} : x > b\}, \quad \{(x,b) \in \mathbb{Z} \times \mathbb{Z} : x = b\}.$$

5.110. Many answers are possible. One is: Define

$$A = \{x \in \mathbb{Z} : x \equiv 0 \pmod 3\},$$
$$B = \{x \in \mathbb{Z} : x \equiv 1 \pmod 3\},$$
$$C = \{x \in \mathbb{Z} : x \equiv 2 \pmod 3\}.$$

Since $\mathbb{Z} = A \cup B \cup C$ while $\varnothing = A \cap B \cap C$, we see that $\{A, B, C\}$ is a partition of $\mathbb{Z}$ with more than one subset.

5.112. *Proof.* Let $\mathbb{R}$ be the universal set and note that $\mathbb{I} = \overline{\mathbb{Q}}$ since $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$. From Theorem 5.40 we see that $\mathbb{Q} \cap \mathbb{I} = \mathbb{Q} \cap \overline{\mathbb{Q}} = \varnothing$ and $\mathbb{Q} \cup \mathbb{I} = \mathbb{Q} \cup \overline{\mathbb{Q}} = \mathbb{R}$, which proves that $\{\mathbb{Q}, \mathbb{I}\}$ is a partition of $\mathbb{R}$. □

5.117. R is a relation on $\mathbb{Z}$ because $R \subseteq \mathbb{Z} \times \mathbb{Z}$.

5.118. Note that $\{(1,2), (345,7), (43,8675309), (11,11)\} \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ so this set is a relation on $\mathbb{Z}^+$.

5.120. (a) T <u>is not</u> reflexive since no person is (strictly) taller than theirself.

(b) N is reflexive since each person has the same name as theirself.

(c) C is reflexive since each person has been to the same city as theirself.

(d) K is not reflexive since a person cannot know theirself (except, perhaps in a philosophical, existential sense).

(e) R is not reflexive since, to be reflexive, a relation containing $\{\text{Barack Obama, George W. Bush}\}$ would also need to contain $\{\text{Barack Obama, Barack Obama}\}$. Since R does not contain this element, it is not reflexive.

5.122. (a) T is not symmetric since if a is taller than b then b cannot be taller than a.

(b) N is symmetric since if a's name starts with the same letter as b's, then obviously b's name starts with the same letter as a's.

(c) C is symmetric: if a and b have been to same city, then it is true to say b and a have been to the same city.

(d) K is not symmetric. I'm pretty sure George Clooney does not know who I am, but I know who he is.

(e) R is not symmetric since R does not also contain $\{\text{George W. Bush, Barack Obama}\}$.

5.124. (a) No, we cannot tell if R is anti-symmetric. It is possible for and anti-symmetric relation to contain ordered pairs of the form (x, y) and (y, x), but only if $x = y$. Ordered pairs like $(1, 1)$ may be found in anti-symmetric relations.

(b) Yes, we can tell that R is not anti-symmetric. If R contains both $(1, 2)$ and $(2, 1)$, it cannot be anti-symmetric. Note that we might call this a *symmetric pair*, but the presence of this pair does not mean that R is symmetric.

5.125. It is the contrapositive of the original definition.

5.126. (a) T is anti-symmetric since if a is taller than b then b cannot be taller than a.

(b) N is not anti-symmetric since if a's name starts with the same letter as b's, then obviously b's name starts with the same letter as a's.

(c) C is not anti-symmetric: if a and b have been to same city, then it is true to say b and a have been to the same city.

(d) K is not anti-symmetric, assuming there are at least two people who know each other.

(e) R is anti-symmetric since R does not also contain $\{\text{George W. Bush, Barack Obama}\}$.

5.127. (a) No, a relation may be neither symmetric nor anti-symmetric. Consider $R = \{(1, 2), (2, 3), (3, 2)\}$. This is not symmetric since it contains $(1, 2)$ but not $(2, 1)$. It is not anti-symmetric since it contains both $(2, 3)$ and $(3, 2)$.

(b) No, see the previous answer.

(c) Yes. For example consider $R = \{(1,1),(2,2),(3,3)\}$. This relation is symmetric since for every $(x, y) \in R$ we find that $(y, x) \in R$. The relation is also anti-symmetric since if $x \neq y$ then (x, y) and (y, x) are not both in R.

5.128. Let $R = \{(1,1),(2,2),(3,3)\}$ be a relation on $\{1,2,3\}$. As argued in the last example, R is both symmetric and anti-symmetric.

5.130. (a) T is transitive since if a is taller than b and b is taller than c then a must be taller than c.

(b) N is transitive since if a's name starts with the same letter as b's and b's name starts with the same letter as c's, then a's name must start with the same letter as c's.

(c) C is not transitive: Suppose a and b have both been to Boston and that b and c have both been to New York City. It may still be possible that a has not been to New York City or that c has not been to Boston, so (a, c) is not necessarily in the relation C.

(d) K is not transitive. Suppose Paul and Will know each other, and Will and Ally know each other. This does not imply that either Paul knows Ally or that Ally knows Paul.

(e) R is transitive. This is perhaps best understood by considering that there is nothing that violates transitivity. Barack Obama is paired with George W. Bush, but George W. Bush is not paired with other person X, so the fact that Barack Obama is not paired with X is not a problem.

5.133. (a) T is not an equivalence relation since it is neither reflexive nor symmetric.

(b) N is an equivalence relation because it is reflexive, symmetric, and transitive.

(c) C is not an equivalence relation since it is not transitive.

(d) K is not an equivalence relation because it is not reflexive, symmetric, or transitive.

(e) R is not an equivalence relation because it is neither reflexive nor symmetric.

5.135. (a) T is not an partial order since it is not reflexive.

(b) N is not an partial order since it is not anti-symmetric.

(c) C is not an partial order since it is neither anti-symmetric nor transitive.

(d) K is not an partial order because it is not reflexive, symmetric, or transitive.

(e) R is not an partial order because it is not reflexive.

5.136. *Proof.* Let sets A, B, and C belong to the collection of sets X. The relation R is **reflexive** since every set A is a subset of itself: $A \subseteq A$. We know that R is **anti-symmetric** since if both $A \subseteq B$ and $B \subseteq A$ then $A = B$ by Theorem 5.44. Finally we see that R is transitive because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ by the definition of the $\subseteq$ operation. □

5.137. (a) R is not reflexive since the ordered pairs $(1,1)$, $(3,3)$, and $(4,4)$ are not present. To be reflexive, *every* ordered pair (a,a) for which $a \in \{1,2,3,4,5\}$ must be present in R.

(b) R is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$. (Other counter examples exist).

(c) R is anti-symmetric since $(1,2) \in R$ but $(2,1) \notin R$, $(1,3) \in R$ but $(3,1) \notin R$, $(1,5) \in R$ but $(5,1) \notin R$, $(2,2) \in R$ and $(2,2) \in R$ but $2 = 2$, $(3,5) \in R$ but $(5,3) \notin R$, and finally $(5,5) \in R$ and $(5,5) \in R$ but $5 = 5$. Alternatively, we could just observe that for every ordered pair (x,y) in R, $x \le y$. Therefore no ordered pairs (y,x) are in R except those for which $x = y$.

(d) R is transitive: $(1,2) \in R$ and $(2,2) \in R$ but we (trivially) see that $(1,2) \in R$. Similarly $(1,3) \in R$ and $(3,5) \in R$ but $(1,5)$ is also in R. Finally, $(1,5) \in R$ and $(5,5) \in R$ but (again trivially) $(1,5) \in R$.

(e) R is not an equivalence relation since it is not reflexive or symmetric.

(f) R is not a partial order since it is not reflexive.

5.139. $((a,b),(a,b))$; bc; da; $((c,d),(a,b))$; symmetric; $ad = bc$; $cf = de$; $c = de/f$; $ad = bde/f$; $af = be$; $((a,b,(e,f))$.

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6 Sequences and Summations

6.1 Sequences

6.3. (a) $x_0 = 2$, (b) $x_1 = -1$, (c) $x_2 = 5$, (d) $x_3 = -7$, (e) $x_4 = 17$

6.4. (a) $x_0 = 2$, $x_1 = \frac{1}{2}$, $x_2 = \frac{5}{4}$, $x_3 = \frac{7}{8}$, $x_5 = \frac{17}{16}$

(b) $x_0 = 2$, $x_1 = 2$, $x_2 = 3$, $x_3 = 7$, $x_4 = 25$

(c) $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{5}$, $x_4 = \frac{1}{25}$, $x_5 = \frac{1}{119}$, $x_6 = \frac{1}{721}$

(d) $x_1 = 2$, $x_2 = \left(\frac{3}{2}\right)^2$, $x_3 = \left(\frac{4}{3}\right)^3$, $x_4 = \left(\frac{5}{4}\right)^4$, $x_5 = \left(\frac{6}{5}\right)^5$

6.7. We note that

$$x_0 = 1,\ x_1 = 5 \cdot 1 = 5,\ x_2 = 5 \cdot 5 = 5^2,\ x_3 = 5 \cdot 5^2 = 5^3, \ldots$$

so it appears that $\boxed{x_n = 5^n, \text{ for } n = 0, 1, 2, \ldots}$

6.8. We note that

$$x_0 = 1,\ x_1 = 1 \cdot 1,\ x_2 = 2 \cdot 1,\ x_3 = 3 \cdot 2 \cdot 1,\ x_4 = 4 \cdot 3 \cdot 2 \cdot 1, \ldots$$

so it appears that $\boxed{x_n = n!, \text{ for } n = 0, 1, 2, \ldots}$

6.9. The examples shown are all correct, but the conclusion reached essentially assumes that $\left\lfloor \frac{1+\sqrt{5}}{2} \times a_{n-1} \right\rfloor = \left\lfloor \frac{1+\sqrt{5}}{2} \right\rfloor \times a_{n-1}$. This happens to be true when a_{n-1} is 2, 4, or 8, but fails to be true when it is 16.

Computing a_5 would have been sufficient to show that $a_n \neq 2^n$, but in other cases one might have to try many examples before finding a counterexample. This once again shows that any number of examples do not prove a pattern.

6.11. The recursive definition of $\{x_n\}$ is $x_0 = 1$, $x_n = 5x_{n-1}$ for $n = 1, 2, \ldots$. To see that $x_n = 5^n$ is a closed form of the same sequence, we first note that $x_0 = 5^0 = 1$ so it works for the initial condition. Now we substitute it into the recursive definition, replacing x_{n-1} with 5^{n-1}:

$$5x_{n-1} = 5(5^{n-1}) = 5^n = x_n$$

Since the closed form works for both the initial condition and satisfies the recurrence relation, we know it expresses the terms in the sequence correctly.

6.12. The recursive definition of $\{x_n\}$ is $x_0 = 1$, $x_n = n \cdot x_{n-1}$ for $n = 1, 2, \ldots$. To see that $x_n = n!$ is a closed form of the same sequence, we first note that $x_0 = 0! = 1$ so it works for the initial condition. Now we substitute it into the recursive definition, replacing x_{n-1} with $(n-1)!$:

$$n \cdot x_{n-1} = n \cdot (n-1)! = n! = x_n$$

Since the closed form works for both the initial condition and satisfies the recurrence relation, we know it expresses the terms in the sequence correctly.

6.13. The analysis and solution offered are correct as long as $n \geq 0$. The definition of `ferzle(n)`, however, allows for $n < 0$ and the proposed solution does not cover this case correctly. When $n < 0$, `ferzle(n)` should return 3.

6.14.

```
int ferzle(int n) {
  // if (n<=0) return 3 else return 2n+3;
  return ( n <= 0 ? 3 : 2*n+3 );
}
```

6.18. Requiring $x_n > x_{n-1}$ is equivalent to requiring $x_n < x_{n+1}$ for some index n. This is because when n is incremented so $n+1 \to n$ the first inequality becomes the second. Both of these indicate that a given term in the sequence is strictly less than the term that immediately succeeds it.

6.20. *Proof.* We show the sequence is strictly increasing by showing that $x_{n+1} - x_n > 0$ for $n = 1, 2, \ldots$.

$$\begin{aligned} x_{n+1} - x_n &= \frac{(n+1)^2+1}{n+1} - \frac{n^2+1}{n} \\ &= \frac{n(n^2+2n+2) - (n+1)(n^2+1)}{n(n+1)} \\ &= \frac{n^3+2n^2+2n-n^3-n^2-n-1}{n(n+1)} \\ &= \frac{n^2+n-1}{n^2+n} \\ &= 1 - \frac{1}{n^2+n} \\ &> 0 \end{aligned}$$

as long as $n > 0$. Thus $x_{n+1} > x_n$ so the sequence is strictly increasing. □

6.21. (a) $x_n = n, n = 0, 1, 2, \ldots$ is strictly increasing since $n+1 > n$.

(b) $x_n = (-1)^n n, n = 0, 1, 2, \ldots$ is non-monotonic since it alternates with $x_0 > x_1$, $x_1 < x_2$, $x_2 > x_3$, $x_3 < x_4$, etc.

(c) $x_n = \frac{1}{n!}, n = 0, 1, 2, \ldots$ is decreasing since $(n+1)! \geq n!$ so $x_{n+1} \leq x_n$. It is not strictly decreasing because $0! = 1!$ so $x_0 = x_1$. However, this sequence is strictly decreasing when $n > 0$.

(d) $x_n = \frac{n}{n+1}, n = 0, 1, 2, \ldots$ is strictly increasing. Notice that

$$x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$$

which shows that $x_{n+1} > x_n$.

(e) $x_n = n^2 - n, n = 1, 2, \ldots$ is strictly increasing. Observe that

$$\begin{aligned} x_{n+1} - x_n &= [(n+1)^2 - (n+1)] - [n^2 - n] \\ &= n^2 + 2n + 1 - n - 1 - n^2 + n \\ &= 2n > 0 \end{aligned}$$

for all $n > 0$. Thus $x_{n+1} > x_n$.

(f) $x_n = n^2 - n, n = 0, 1, 2, \ldots$ is increasing. We just showed that this sequence is strictly increasing when $n > 0$ but notice that $x_0 = 0^2 - 0 = 0$ and $x_1 = 1^2 - 1 = 0$ so this sequence is not strictly increasing.

(g) $x_n = (-1)^n, n = 0, 1, 2, \ldots$ is non-monotonic. The values of the sequence alternate between $+1$ and -1.

(h) $x_n = 1 - \frac{1}{2^n}, n = 0, 1, 2, \ldots$ is strictly increasing since $x_{n+1} - x_n = \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} > 0$.

(i) $x_n = 1 + \frac{1}{2^n}, n = 0, 1, 2, \ldots$ is strictly decreasing since $x_{n+1} - x_n = \frac{1}{2^{n+1}} - \frac{1}{2^n} = -\frac{1}{2^{n+1}} < 0$.

6.24. In this progression $a = -\frac{2}{3^{17}}$ and $r = \frac{2}{3^{16}} \cdot \frac{3^{17}}{-2} = -3$. The 17-th term is

$$-\frac{2}{3^{17}} \cdot (-3)^{16} = -\frac{2}{3}.$$

6.26. We are told $ar^5 = 20$ and $ar^9 = 320$. Since $ar \neq 0$ we can compute

$$\frac{ar^9}{ar^5} = r^4 = \frac{320}{20} = 16$$

so $|r| = 2$. In this case $|a| = |20/r^5| = 20/32 = 5/8$. The absolute value of the third term is

$$|ar^2| = \frac{5}{8} \cdot 2^2 = \frac{5}{2}.$$

6.30. (a) The difference between consecutive terms is 7 and the starting term is 2 so the 8-th term will be $2 + (8-1) \cdot 7 = 51$. Thus the 'correct' answer is (d).

(b) The word 'correct' is in quotes since the question implies that the pattern shown in the listed terms continues. The answer is correct assuming the sequence is an arithmetic progression.

6.31. (a) The sequence in 6.7 is geometric since each term is generated from the previous term by multiplying by 5 (i.e. 5 is the common ratio).

(b) The sequence in 6.8 is neither geometric nor arithmetic. There is no common ratio nor is their a common difference.

(c) The sequence from 6.13 generated by `ferzle(n)` when $n \geq 0$ is arithmetic. The initial term is 3 and common difference is 2.

6.2 Sums and Products

6.34. $\sum_{k=0}^{100} y^k$

6.36. $\sum_{k=0}^{50} y^{2k}$

6.38. (a) 2, (b) 11, (c) 100, (d) 101

6.43. (a) 10, (b) 2200

6.46. (a) 10, (b) 2200, (c) 900, (d) 909

6.47. There are $75 - 25 + 1 = 51$ terms in the sum, each of which is 10. The sum should be $51 \times 10 = 510$.

6.50. (a) $\sum_{k=1}^{20} k = \frac{20(20+1)}{2} = 210$

(b) $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050$

(c) $\sum_{k=1}^{1000} k = \frac{1000(1000+1)}{2} = 500500$

6.51. *Solution 1:* This is incorrect since it uses the formula $(n-1)n/2$ rather than $n(n+1)/2$.

Solution 2: This is incorrect since k is factored out of the sum even though k is not constant. Only constant factors may be factored out of a sum.

6.52. It is true that $\sum_{k=0}^{n} k = \sum_{k=1}^{n} = \frac{n(n+1)}{2}$. While the first sum has $n+1$ terms and the second sum has only n terms, the "extra" term in the first sum corresponds to $k = 0$ and so is zero.

6.55. $\sum_{i=1}^{100} 2 - i = \sum_{i=1}^{100} 2 - \sum_{i=1}^{100} i = 200 - \frac{100 \cdot 101}{2} = 200 - 5050 = -4850.$

6.57. The sum of the first n odd positive integers is n^2.

Proof. The sum of the first n odd positive integers is equal to the sum of the the first $2n$ positive integers minus the sum of the first n even integers. Thus

$$\begin{aligned}\sum_{k=1}^{2n} k - \sum_{k=1}^{n} 2k &= \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} \\ &= (2n^2 + n) - (n^2 + n) \\ &= n^2.\end{aligned}$$

□

6.59. $\sum_{k=10}^{20} k = \sum_{k=1}^{20} k - \sum_{k=1}^{9} k = \frac{20 \cdot 21}{2} - \frac{9 \cdot 10}{2} = 210 - 45 = 165.$

6.60. *Solution 1:* This computation is nearly correct, but the summation that is subtracted should go from $k = 1$ to 29. The answer computed here is really $\sum_{k=31}^{100} k$.

Solution 2: This computation makes the same set-up error as the previous solution attempt, but also incorrectly computes $\sum_{k=1}^{n}$ as $(n-1)n/2$.

Solution 3: This solution is correct as written.

6.61. The computation is incorrect for two reasons. First, the second sum should be $\sum_{k=1}^{29} k$. Second, $\sum_{k=1}^{30} k$ should be $30 \cdot 31/2$, not $29 \cdot 30/2$. However, the answer is correct since the second error made actually corrects for the first error; i.e., the computation $29 \cdot 30/2$ correctly calculates $\sum_{k=1}^{29} k$.

6.63. A cheeky answer would be "because it is correct for the sum to have a lower index of 2." However, the intent of the question is probably to point out that since both k and $k - 1$ appear in the denominator, neither can be 0, so $k \neq 0$ and $k \neq 1$.

6.64.

$$\begin{aligned}\sum_{k=1}^{n}(k^3+k) &= \sum_{k=1}^{n}k^3+\sum_{k=1}^{n}k \\ &= \frac{n^2(n+1)^2}{4}+\frac{n(n+1)}{2} \\ &= \frac{n(n+1)\cdot n(n+1)}{4}+\frac{n(n+1)\cdot 2}{4} \\ &= \frac{n(n+1)(n^2+n+2)}{4}\end{aligned}$$

6.66. (a) $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{i}1=\sum_{i=1}^{n}i=\frac{n(n+1)}{2}.$

(b) $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{i}j=\sum_{i=1}^{n}\frac{i(i+1)}{2}=\frac{1}{2}\left[\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i\right]=\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$
$\displaystyle=\frac{n(n+1)(n+2)}{6}$

(c) $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}ij=\sum_{i=1}^{n}\left(i\sum_{j=1}^{n}j\right)=\sum_{i=1}^{n}i\frac{n(n+1)}{2}=\frac{n(n+1)}{2}\sum_{i=1}^{n}i=\frac{n^2(n+1)^2}{4}$

6.69. (a) $\displaystyle 1+3+3^2+3^3+\cdots+3^{49}=\frac{3^{50}-1}{3-1}=\frac{3^{50}-1}{2}$

(b) $1+y+y^2+\cdots+y^{100}=\frac{1-y^{101}}{1-y}$

(c) $\displaystyle 1-y+y^2-y^3+\cdots-y^{99}+y^{100}=\frac{1-(-y)^{101}}{1+y}$

(d) $\displaystyle 1+y^2+y^4+\cdots+y^{100}=1+y^2+(y^2)^2+\cdots+(y^2)^{50}=\frac{(y^2)^{51}-1}{y^2-1}=\frac{1-y^{102}}{1-y^2}$

6.72. $x^5-1=(x-1)(x^4+x^3+x^2+x+1)$

6.73. $2^1+2^2+2^3+2^4+\cdots+2^{n+1}$; 2^0; 2^{n+1}; -2^0+2^{n+1}.

6.75. $\displaystyle a\sum_{k=0}^{n}r^k$; $\displaystyle a\frac{1-r^{n+1}}{1-r}$.

6.76. *Proof.* Suppose $r \neq 1$ and a are real numbers and let $S = \sum_{k=0}^{n} ar^k$. Then

$$\begin{aligned} rS - S &= \sum k = 1^{n+1} ar^k - \sum_{k=0}^{n} ar^k \\ (r-1)S &= \left(ar + ar^2 + \cdots + ar^n + ar^{n+1}\right) - \left(a + ar + ar^2 + \cdots + ar^n\right) \\ &= ar^{n+1} - a \end{aligned}$$

Dividing both sides by $r - 1$ yields $S = \dfrac{ar^{n+1} - a}{r - 1}$. Therefore

$$\sum_{k=0}^{n} ar^k = \frac{ar^{n+1} - a}{r - 1} \qquad \text{or} \qquad \sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r}.$$

□

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7 Algorithm Analysis

7.1 Asymptotic Notation

7.6. (a) $-3n \leq 0$ when $n > 1$; (b) $20 < 20n^2$ when $n > 1$.

7.7. While it is true that $10 \leq 10n^2$ when $n \geq 1$, it is not true that $-12n \leq -12n^2$ for these same values of n. The approach used here would work if all terms were summed.

7.8. (a) Yes, $c = 50$ could be used; (b) No, $c = 2$ would not work since the coefficient of n^2 is 5 in the expression we are trying to bound; (c) Yes, we could use $n_0 = 100$ since this value is larger than the $n_0 = 1$ used in Example 7.5; (d) No, $n_0 = 0$ would not work since this would allow $n = 0$ and it is not true that $20 \leq 0$.

7.9. If $n \geq 1$ we have

$$\begin{aligned} 5n^5 - 4n^4 + 3n^3 - 2n^2 + n &\leq 5n^5 + 3n^3 + n && \text{since } -4n^4 - 2n^2 \leq 0 \\ &\leq 5n^5 + 3n^5 + n^5 && \text{since } n^3 \leq n^5 \text{ and } n \leq n^5 \\ &= 9n^5 \end{aligned}$$

Since we have shown that $5n^5 - 4n^4 + 3n^3 - 2n^2 + n \leq 9n^5$ for all $n \geq 1$, we know that $5n^5 - 4n^4 + 3n^3 - 2n^2 + n = O(n^5)$.

7.10. $n_0 = 1$, $c = 9$.

7.13. If $n \geq 1$ we know $n + 1 \geq 0$ so $4n^2 + n + 1 \geq 4n^2$. Since $4n^2 + n + 1 \geq 4n^2$ for all $n \geq 1$, we know that $4n^2 + n + 1 = \Omega(n^2)$.

7.14. $n_0 = 1$, $c = 4$.

7.17. If need to satisfy both $n \geq 0$ and $n \geq 1$, it is sufficient to require $n \geq 1$, since this restriction satisfies both requirements.

7.22. Since $g(n)$ appears in the denominator, it must be nonzero. As we are interested in the limit as $n \to \infty$ it is not a problem if $g(n) = 0$ for some particular value(s) of n, only that $g(n) \neq 0$ for all values of n larger than some integer; this is what it means to say g is eventually non-zero.

7.23. If big-O notation is equivalent to $\leq$ in some ways, then o would be equivalent to $<$ and ω would be equivalent to $>$.

7.24. (a) If $f(n) = \Theta(g(n))$, it is not possible that $f(n) = o(g(n))$ since $f(n)$ cannot grow at the same asymptotic rate as $g(n)$ and at a rate asymptotically slower than $g(n)$.

(b) If $f(n) = O(g(n))$, it is possible that $f(n) = o(g(n))$ since the first statement says f grows no faster than $g(n)$ and the second says f grows more slowly than $g(n)$.

(c) If $f(n) = O(g(n))$, it is not certain that $f(n) = o(g(n))$ since the second statement says f grows more slowly than $g(n)$ while the first says f can grow at the same rate as $g(n)$.

(d) If $f(n) = o(g(n))$, it is possible that $f(n) = O(g(n))$ since the second statement says f grows no faster than $g(n)$ and the first says f grows more slowly than $g(n)$.

7.25. *Solution 1:* While not strictly incorrect, this "proof" makes many rather loose and unsubstantiated claims and no attention is given to the values of n for which the result shown applies. It is possible to show that $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 = \Theta(n^k)$, but this should be done carefully with pair of inequalities. This is also a stronger result than we are asked to prove, and merely showing that the expression has an $O(n^k)$ bound will involve less work.

Solution 2: This is a good proof!

7.26. We can't say much, only that g *could* grow faster than f, but this does not have to be true.

7.27. (a) *False.* Note that $n = O(n^2)$ but n does not grow faster than n^2.

(b) *False.* By definition both f and g must grow at the same asymptotic rate.

(c) *False.* The function f can grow at the same rate as g but it could also grow more slowly.

(d) *False.* We know that f grows at least as fast as g, but they could grow at the same rate.

(e) *False.* Note that $n = O(n^2)$ but $n \neq \Omega(n^2)$.

(f) *True.* If $f(n) = \Theta(g(n))$ then both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

(g) *False.* The function f could grow at the same rate as g but it could also grow more slowly so that it is not in $\Theta(g(n))$. For example $n = O(n^2)$ but $n \neq \Theta(n^2)$.

(h) *False.* Consider that $n = O(n^2)$ while $n^2 \neq O(n)$.

7.37. $c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$.; $\frac{1}{c_2} f(n)$; $\frac{1}{c_1} f(n)$; $c_3 h(n) \le g(n) \le c_4 h(n)$ for all $n \ge n_1$.; c_2; $c_2 c_4$; $\max\{n_0, n_1\}$.; $c_1 c_3 h(n)$; $c_2 c_4 h(n)$; $\Theta(h(n))$ Θ; transitive.;

7.39. (a) *True.* If f is bounded from above by g then g is bounded from below by f.

(b) *True.* If f and g grow at the same rate then f is bounded from above and from below by g.

(c) *True.* If f_1 and f_2 are bounded from above by g_1 and g_2 respectively, then $f_1 + f_2$ must be bounded from above by the larger of g_1 and g_2.

(d) *False.* Note that $n = O(n^2)$ but $n \neq \Theta(n^2)$.

(e) *False.* Note that $n = O(n^2)$ but $n^2 \neq O(n)$.

(f) *True.* If f is bounded above by g, then g is bounded from below by f.

(g) *True.* This is a theorem that follows immediately from the definition of Θ.

(h) *True.* If f is bounded from above by g and g is bounded from above by h, then f must be bounded from above by h.

7.42. *Proof.* We need to find positive constants c_1, c_2, and n_0 such that

$$\underline{\quad c_1 n^2 \quad} \le \frac{1}{2} n^2 - 3n \le \underline{\quad c_2 n^2 \quad} \text{ for all } n \ge n_0$$

Dividing by n^2, we get

$$c_1 \le \underline{\quad \frac{1}{2} - \frac{3}{n} \quad} \le c_2.$$

Notice that if $n \ge 10$,

$$\frac{1}{2} - \frac{3}{n} \ge \frac{1}{2} - \frac{3}{10} = \underline{\quad \frac{5-3}{10} = \frac{1}{5} \quad},$$

so we can choose $c_1 = 1/5$. If $n \geq 10$, we also have that $\frac{1}{2} - \frac{3}{n} \leq \frac{1}{2}$, so we can choose $c_2 = 1/2$. Thus, we have shown that

$$\underline{\frac{1}{5}c_1 n^2} \leq \frac{1}{2}n^2 - 3n \leq \underline{\frac{1}{2}c_2 n^2} \text{ for all } n \geq \underline{\ 10\ }.$$

Therefore, $\frac{1}{2}n^2 - 3n = \Theta(n^2)$. □

7.43. If $n \geq 10$ then $\frac{1}{n} \leq \frac{1}{10}$. Multiplying both sides of this inequality by -3 we have $-\frac{3}{n} \geq -\frac{3}{10}$. Finally, adding $1/2$ to both sides gives $\frac{1}{2} - \frac{3}{n} \geq \frac{1}{2} - \frac{3}{10}$.

7.45. (a) Theorem 7.18

(b) No, knowing $f(n) = O(g(n))$ does not imply $f(n) = \Theta(g(n))$. To be able able to draw this conclusion we would also need to know that $f(n) = \Omega(g(n))$.

7.46. *Proof.* Recall that $n!$ is defined for all integers $n \geq 0$ to be $n! = n(n-1)(n-2)\cdots(2)(1)$. There are n factors, each of the form $n-k$ for $k = 0, 1, \ldots, n-1$, present when $n > 0$. For each factor we know $n - k \leq n$. Thus

$$n! = n(n-1)(n-2)\cdots(2)(1) \leq n^n$$

so we see that $n! = O(n^n)$. □

7.49. *Proof.* Suppose $f(x) = O(g(x))$ and $g(x) = O(h(x))$. Then there exist constants c_1, c_2, n_1, and n_2 such that

$$f(x) \leq c_1 g(x) \text{ for all } x \geq n_1 \quad \text{and} \quad g(x) \leq c_2 h(x) \text{ for all } x \geq n_2.$$

Clearly both these inequalities are also true for $x \geq n_0$ if $n_0 = \max\{n_1, n_2\}$. Replacing $g(n)$ in the first inequality using the second, we have

$$f(x) \leq c_1 c_2 h(x) \text{ for all } x \geq n_0,$$

or, if $c_0 = c_1 c_2$,

$$f(x) \leq c_0 h(x) \text{ for all } x \geq n_0,$$

proving that $f(x) = O(h(x))$. □

7.53. (a) $\lim_{n\to\infty} \log_{10} n = \infty$

(b) $\lim_{n\to\infty} n^3 = \infty$

(c) $\lim_{n\to\infty} 3^n = \infty$

(d) $\lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$

(e) $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

(f) $\lim_{n\to\infty} n^{-1} = 0$

(g) $\lim_{n\to\infty} 8675309 = 8675309$

7.57. Theorem 7.51 was used to conclude that $\lim_{n\to\infty} n^2 = \infty$ and Theorem 7.55 was then used to conclude that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

7.58. *Proof.* Notice that

$$\lim_{x\to\infty} \frac{3x^3}{x^2} = \lim_{x\to\infty} 3x = \infty$$

so $3x^3 = \Omega(x^2)$ by Theorem 7.50 (case 2). □

7.63. *Proof.* Notice that

$$\lim_{n\to\infty} \frac{n(n+1)/2}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}$$

so $n(n+1)/2 = \Theta(n^2)$ by Theorem 7.50 (case 3). □

7.64. (a) *Proof.* Notice that

$$\lim_{x\to\infty} \frac{2^x}{3^x} = \lim_{x\to\infty} \left(\frac{2}{3}\right)^x = 0$$

so $2^x = O(3^x)$ by Theorem 7.50 (case 1). □

(b) *Proof.* We need to show that constants c and n_0 can be found so that $2^x \leq c \cdot 3^x$ for all $x \geq n_0$. We note that

$$\frac{2^x}{3^x} = \left(\frac{2}{3}\right)^x \leq 1 \text{ for all } x \geq 1.$$

Therefore $2^x \leq 1 \cdot 3^x$ for all $x \geq 1$. If we set $c = 1$ and $n_0 = 1$ then we see that $2^x = O(3^x)$ by the definition of big-O. □

7.70. *Proof 1:* The writer writes $7^x - 5^x > 0$, where they should say $7^x/5^x > 1$. It's not the *difference* between the two functions that matters, but rather their relative growth rates. The argument is also rather loose in general.

Proof 2: The writer makes the mistake of applying the log function to both numerator and denominator. In general it is not true that $a/b = \log a/\log b$.

Proof 3: This is essentially correct as it uses sound reasoning. It implicitly uses Theorems 7.50 and 7.51 to draw the conclusion, but does not explicitly address why the bound is not tight.

7.72. (a) *Proof.* Notice that, for $n \geq 2$,

$$\begin{aligned} n\ln(n^2+1) + n^2\ln n &\leq n\ln(n^2+n^2) + n^2\ln n && \text{since } 1 \leq n^2 \\ &= n\ln(2n^2) + n^2\ln n \\ &= n\ln 2 + n\ln n^2 + n^2\ln n \\ &\leq n\ln n + 2n\ln n + n^2\ln n && \text{since } 2 \leq n \\ &= 3n\ln n + n^2\ln n \\ &\leq 3n^2\ln n + n^2\ln n && \text{since } n \leq n^2 \\ &= 4n^2\ln n \end{aligned}$$

Therefore $n\ln(n^2+1) + n^2\ln n = O(n^2\ln n)$. □

(b) *Proof.* By inspection we notice that for large n the first term resembles $n\ln n^2$, which equals $2n\ln n$, while the second term is $n^2\ln n$. Since $2n < n^2$ for large n, we can conjecture that the upper bound will be $O(n^2\ln n)$. Notice

$$\begin{aligned} \lim_{n\to\infty} \frac{n\ln(n^2+1) + n^2\ln n}{n^2\ln n} &= \lim_{n\to\infty} \frac{\ln(n^2+1)}{n\ln n} + 1 \\ &= \lim_{n\to\infty} \frac{2n/(n^2+1)}{\ln n + 1} + 1 && \text{by l'Hôpital's Rule} \\ &= \frac{\lim_{n\to\infty}(2n/(n^2+1))}{\lim_{n\to\infty}(\ln n + 1)} + 1 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

This result shows that $n\ln(n^2+1) + n^2\ln n = \Theta(n^2\ln n)$ by Theorem 7.50. Thus $n^2\ln n$ is not only an upper bound, it is also a tight bound. □

7.74. *Proof.* We conjecture the bound will be n^{10} since that is the dominant term in $(n^2-1)^5$.

$$\begin{aligned} \lim_{n\to\infty} \frac{(n^2-1)^5}{n^{10}} &= \lim_{n\to\infty} \left(\frac{n^2-1}{n^2}\right)^5 \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^5 \\ &= \left(1 + \lim_{n\to\infty} \frac{1}{n^2}\right)^5 \\ &= (1+0)^5 \\ &= 1 \end{aligned}$$

Thus, by Theorem 7.50, we know $(n^2-1)^5 = \Theta(n^{10})$. □

7.75. *Proof.* We note that $2^{n+1} + 5^{n-1} = 2 \cdot 2^n + 5^n/5$ so that it appears that the bound will be 5^n.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^{n+1} + 5^{n-1}}{5^n} &= \lim_{n\to\infty} \frac{2 \cdot 2^n + 5^n/5}{5^n} \\
&= \lim_{n\to\infty} \left[2 \cdot \left(\frac{2}{5}\right)^n + \frac{1}{5} \cdot \left(\frac{5}{5}\right)^n \right] \\
&= 2 \cdot 0 + \frac{1}{5} \cdot 1 \\
&= \frac{1}{5}
\end{aligned}$$

By Theorem 7.50 we see that $2^{n+1} + 5^{n-1} = \Theta(5^n)$. □

7.2 Asymptotic Notation

7.78. *Proof.* Suppose that $a < b$ are real numbers. Notice that

$$\lim_{n\to\infty} \frac{n^a}{n^b} = \lim_{n\to\infty} n^{a-b} = 0$$

since $a - b < 0$. By Theorem 7.50 we know that $n^a = o(n^b)$. □

7.81. *Proof.* Suppose that $0 < a < b$ are real numbers. Notice that

$$\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n = 0$$

since $0 < a/b < 1$. By Theorem 7.50 we know that $n^a = o(n^b)$. □

7.86. (a) False; (b) True; (c) False; (d) True; (e) True; (f) False.

7.89. *Proof.* Let $b > 0$ and $c > 0$ be real numbers. Notice that $\lim_{n\to\infty} \frac{\log_c(n)}{n^b}$ is of the appropriate form for l'Hôpital's Rule since the numerator and denominator both grow without bound as $n \to \infty$. Recall that

$$\frac{d}{dx} \log_c(x) = \frac{d}{dx} \frac{\ln(x)}{\ln(c)} = \frac{1}{x \ln(c)}$$

so

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log_c(n)}{n^b} &= \lim_{n\to\infty} \frac{1}{(n \ln c)(bn^{b-1})} = \lim_{n\to\infty} \frac{1}{bn^b \ln c} \\
&= \frac{1}{b \ln c} \left(\lim_{n\to\infty} \frac{1}{n^b} \right) = \frac{1}{b \ln c} \cdot 0 = 0
\end{aligned}$$

since n^b will grow without bound. By Theorem 7.50, $\log_c(n) = o(n^b)$. □

7.94. (a) Θ; (b) o; (c) Θ; (d) o; (e) Θ; (f) ω; (g) o; (h) o; (i) o; (j) o;

7.96. (a) $\Theta(n^7)$; (b) $\Theta(n^8)$; (c) $\Theta(n^2)$; (d) $\Theta(3^n)$; (d) $\Theta(2^n)$; (f) $\Theta(n^2)$; (g) $\Theta(n^{.000001})$; (h) $\Theta(n^n)$;

7.97. Here $f(x) \ll g(x)$ means $f(x) = o(g(x))$ and commas separate functions with the same growth rate.

$$\begin{aligned} &10000 \ll \log x, \log x^{300} \ll \log^{300} x \\ &\ll x^{.000001} \ll x, \log 2^x \ll x \log x \ll x^{\log 3} \\ &\ll x^2 \ll x^5 \ll 2^x \ll 3^x \end{aligned}$$

7.3 Algorithm Analysis

7.98. Wall-clock time and CPU time are not the same since different CPUs run at different rates, and a given CPU may also be task switching between multiple jobs so that wall-clock time may be longer than the time spent by the CPU on a job.

7.99. There are two main reasons why we cannot be sure that Stu's algorithm is more efficient. First, the computers Stu and Sue used may run at different rates. Second, We don't know what problem size was used and it is possible that Sue's algorithm will prove to be more efficient as the problem size grows.

7.100. Wall-clock time may not be a reliable indicator of performance if the computer is not dedicated to running a single job as the operating system may interrupt the job to run others. Depending on how this scheduling is done and the overall load on the computer, wall-clock time can be quite variable.

7.101. In general the CPU-time of jobs on the same computer can be used to give a reliable measure of performance since this is a measure of how much time the CPU actually spent working on the job.

7.103. If the input matrix is $n \times m$ then the size of the input would be nm.

7.109. For our analysis we will focus on the `max = max(max,a[i])` assignment and count this as one instruction. In this example the best case, worst case, and average case complexities are all the same. This is because the algorithm always makes n comparisons regardless of the contents of the array. Thus, the complexity for all three cases is $\Theta(n)$.

7.112. In this algorithm the body of the inner loop is always executed $k = 50$ times, so if we count the assignment as a single operation then this loop always does 50 operations. The outer loop is done n times, so there are $50n$ operations in total, which is $\Theta(n)$. This is the best, worst, and average complexity for this algorithm.

7.113. We consider the assignment instruction inside the inner loop as a single instruction and note that the inner loop is done n^2 times. The outer loop is done n times, so the assignment instruction is done $n \cdot n^2 = n^3$ times. Thus, the worst-case complexity is $\Theta(n^3)$.

7.116. (a) `AreaTrapezoid` has constant complexity since it always carries out the same number of instructions regardless of the input.

(b) `factorial` has complexity $\Theta(n)$ (the assignment in the inner loop is done n times) and so does not have constant complexity.

(c) `absoluteValue` has constant complexity since it always carries out the same number of instructions regardless of the input.

7.121. The complexity is only $\Theta(n)$ because the body of the inner loop is always executed six times. If the `sum=sum+i` line is counted as one instruction, this algorithm does $6n$ instructions and so has complexity $\Theta(n)$.

7.126. (a) The `factorial` algorithm has complexity $\Theta(n)$ (the assignment in the inner loop is done n times) and so does not have quadratic complexity.

(b) Since the proposed algorithm essentially does a linear search through a dataset of size n^2, this algorithm will have complexity $\Theta(n^2)$.

7.133. As noted, in the best-case the code inside the conditional statement never executes. The complexity is still $\Theta(n^2)$ because at most this only reduces the number of operations done by a constant factor and $\Theta(cn^2) = \Theta(n^2)$.

7.136. When a linked list is used as the underlying data structure, both `set(i,x)` and `get(i)` will have complexity $\Theta(i)$ since a linear traversal from the start of the list must be performed to find the desired location in the list.

7.138. In some answers below there are two answers given. The first assumes that the number of elements in the structure is stored or can be computed in $\Theta(1)$ while the second assumes that the size must be computed.

Stack	array	linked list
push	$\Theta(1)$	$\Theta(1)$
pop	$\Theta(1)$	$\Theta(1)$
peek	$\Theta(1)$	$\Theta(1)$
size	$\Theta(1)$	$\Theta(1)$ or $\Theta(n)$
isEmpty	$\Theta(1)$	$\Theta(1)$

7.139. The note in 7.138 applies here as well.

Queue	array	linked list	circular array
enqueue	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
dequeue	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
first	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
size	$\Theta(1)$	$\Theta(1)$ or $\Theta(n)$	$\Theta(1)$
isEmpty	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

7.140. The note in 7.138 applies here as well. Additionally, we assume the list has both a head and tail pointer.

List	array	linked list
addToFront	$\Theta(n)$	$\Theta(1)$
addToEnd	$\Theta(1)$	$\Theta(1)$
removeFirst	$\Theta(n)$	$\Theta(1)$
contains	$\Theta(n)$	$\Theta(n)$
size	$\Theta(1)$	$\Theta(1)$ or $\Theta(n)$
isEmpty	$\Theta(1)$	$\Theta(1)$

7.141. Suppose h where $h \leq n$ is the height of the tree. When the tree is balanced $h = \log_2 n$.

BST	unbalanced	balanced
insert/add	$\Theta(h)$	$\Theta(\log n)$
delete/remove	$\Theta(h)$	$\Theta(\log n)$
search/contains	$\Theta(h)$	$\Theta(\log n)$
maximum	$\Theta(h)$	$\Theta(\log n)$
successor	$\Theta(h)$	$\Theta(\log n)$

7.142.

Hash Table	average	worst
insert/add	$\Theta(1)$	$\Theta(n)$
delete/remove	$\Theta(1)$	$\Theta(n)$
search/contains	$\Theta(1)$	$\Theta(n)$

Solution 1: Just because the function evaluates an exponential, does not mean it has exponential complexity.

Solution 2: The answer is not well formed since it does not use big-O notation. Also, if we assume the writer meant to say $O(ni)$, they have used i as part of there answer even though only n should be used.

Solution 3: The end result here is correct, but the reasoning is not quite right. The loops are nested, and the final value of i is $n-1$. This means the number of operations done by `power(a,i)` over all invocations is $(n-1)n/2$. Since the constant divisor does not matter in asymptotic complexity we see that this is $O(n^2)$. Just because final result matches the correct result does not make this problem correct. The reasoning is not and so the overall answer is not correct.

7.144. The number of operations done in the inner loop is $1 + i$ if we count the addition necessary to update `sum`. Thus, the operation done are

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

thus the worst-case complexity is $O(n^2)$. This is also the best and average case complexity.

7.145.

```
double addPowers(double a, int n)
{
  if (a == 1) return n;

  double sum = 1;
  for (int i = 1; i < n; i++)
  {
    sum = sum * a + 1;
  }
  return sum;
}
```

The loop body contains two operations and it done $n-1$ times. The operation count is $2(n-1$ which is $\Theta(n)$.

7.146. We offer two non-loop solutions. The first uses recursion instead of a loop.

```
double addPowers(double a, int n)
{
  if (a == 1) return n;
  if (n == 0) return 0;
  else return a * addPowers(a, n-1) + 1;
  }
}
```

Counting operations is similar to the loop version. There are n function calls after the original one and on $n-1$ of these there is a multiplication and an addition. Thus there are $2(n-1)$ operations done. Here we have neglected the fact that a function call is most likely a more expensive operation than a multiplication or addition, so although the code has the same actual operation count as the loop-based codes, it is probably worse because of the repeated function call overhead.

For a more efficient no-loop version, recall the partial sum of a geometric series is given by

$$a + ar^1 + ar^2 + \cdots + ar^n = \frac{a(1 - r^{n+1})}{1 - r}.$$

Thus we can code the function as

```
double addPowers(double a, int n)
{
  if (a == 1) return n;
  else return (1 - power(a,n)) / (1-a);
}
```

This is $\Theta(n)$ since `power(a,n)` requires n operations.

7.148. *Solution 1:* This has nearly the same operation count as the code in Example 7.147 but computes the sum from m to $n-1$, so it does not correctly compute the sum. The first loop takes about $1 + 4n$ operations while the second takes about $1 + 4m$ operations. With the first and last statements the total operation count is $1 + 4n + 1 + 4m + 2 = 4 + 4(n + m) = 4(1 + n + m)$. This is $\Theta(n + m)$.

Solution 2: This also computes the sum from m to $n - 1$, so it is not correct. The loop body is executed $n - m$ times so the operation count, including the first and last operations, is $2 + 4(n - m) = \Theta(n - m)$.

Solution 3: This solution requires 8 operations and so is $\Theta(1)$. Unfortunately it is also not correct as it once again computes the sum from m to $n - 1$.

7.149.

```
int sumFromMToN(int m, int n)
{
  return (n*(n+1) - (m-1)*m) / 2;
}
```

This is correct since

$$\sum_{i=m}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{m-1} i = \frac{n(n+1)}{2} - \frac{(m-1)m}{2} = \frac{n(n+1) - (m-1)m}{2}.$$

The function performs two multiplications, two additions, one division, one subtraction, and one function return for a total of 7 operations. This is $\Theta(1)$.

7.152. (a) If `l12` is a LinkedList with m elements the call `c.contains(elementData[r])` takes $\Theta(m)$ time, which is the same for the ArrayList example, so the complexity is $\Theta(nm)$.

(b) If `hs2` is a HashSet with m elements the call `c.contains(elementData[r])` takes $\Theta(1)$ time in the average case but $\Theta(m)$ time in the worst-case. Thus the average complexity is $\Theta(1 \cdot n + n) = \Theta(n)$, while the worst-case complexity is again $\Theta(nm)$.

7.154. (a) If `ll2` is a LinkedList with m elements the call to `c.contains` takes $\Theta(m)$ time. The total worst-case complexity for the loop body is is $\Theta(\log n + m)$. The loop executes n times so the overall worst-case complexity is $\Theta(n(\log n + m))$.

(b) If `ts2` is a TreeSet with m elements the call to `c.contains` takes $\Theta(\log m)$ time. The total worst-case complexity for the loop body is is $\Theta(\log n + \log m) = \Theta(\log nm)$. The loop executes n times so the overall worst-case complexity is $\Theta(n(\log nm))$.

7.156.

n		$\lfloor n/2 \rfloor$	
decimal	binary	decimal	binary
12	1100	6	110
13	1101	6	110
32	100000	16	10000
33	100001	16	10000
118	1110110	59	111011
119	1110111	59	111011

7.157. To obtain the binary representation of $\lfloor n/2 \rfloor$, the binary representation of n is shifted to the right one place, discarding the least-significant bit.

---

8 Recursion, Recurrences, and Mathematical Induction

8.1 Mathematical Induction

8.2. (a) **N** - If the domain of the propositional function is $\mathbb{Z}$ then there is no "starting point."

(b) **Y**, but really **N** - Since domain of the propositional function is $\mathbb{Z}^+$ induction might work here. However, the statement is false so we cannot use induction (or any other technique) to prove it. Consider the positive integer 1, which cannot be written as the sum of two positive integers.

(c) **Y** - Again, we have $\mathbb{Z}^+$ as the domain, so this is ordered in a way that lends itself to induction.

(d) **Y** - Ditto (this is a classic induction problem)

(e) **N** - Domain has no starting point.

8.4. modus ponens

8.5. If $P(5)$ is true and $P(k) \to P(k+1)$ when $k \geq 1$, we can conclude that $P(6)$ is also true.

8.6. If $P(a)$ is true and $P(k)$ being true implies $P(k+1)$ is true whenever $k \geq a$, then $P(n)$ is true for all $n \geq a$.

8.7. Oh yeah!

8.10. For every integer $n \geq 1$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof. **Proof:** Let $P(k)$ be the statement "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$." We need to show that $P(n)$ is true for all $n \geq 1$.

Base Case: When $k = 1$ we have $\sum_{i=1}^{1} i = 1 = \underline{\frac{1(1+1)}{2} = 1}$. Therefore, $\underline{P(1) \text{ is true}}$.

Inductive Hypothesis: Let $k \geq 1$ and assume that $\underline{P(k) \text{ is true}}$. That is, assume that $\underline{\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \text{ when } k \geq 1}$.

[This is not part of the proof, but it will help us see what's next. our goal in the next step is to prove that $\underline{P(k+1)}$ is true. That is, we need to show that $\underline{\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}}$.]

Inductive Step: Notice that

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \underline{\sum_{i=1}^{k} i} + (k+1) \\
&= \underline{\frac{k(k+1)}{2}} + (k+1) \qquad \text{(by the inductive hypothesis)} \\
&= (k+1)\underline{\left(\frac{k}{2}+1\right)} \\
&= \underline{\frac{(k+1)(k+2)}{2}}
\end{aligned}$$

Thus $\underline{P(k+1) \text{ is true.}}$

Summary: We showed that $P(1)$ is true and that whenever $k \geq 1$, $P(k) \to P(k+1)$, therefore $P(n)$ is true for $n \geq 1$ by induction. □

8.11. (a) $\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$

(b) $\sum_{i=1}^{k+1} i \cdot i! = (k+2)! - 1$

(c) $\sum_{i=1}^{k} i \cdot i!$

(d) $(k+1)! - 1$

(e) $\sum_{i=1}^{k+1} i \cdot i!$

(f) $(k+2)! - 1$

8.14. Prove that for all $n \geq 1$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Let $P(k)$ be the statement "$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$." We need to prove that $P(n)$ is true for all $n \geq 1$.

Base Case: When $k = 1$ we have $\sum_{i=1}^{1} i^2 = 1^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1$. Therefore $P(1)$ is true.

Inductive Hypothesis: Let $k \geq 1$ and assume that $P(k)$ is true. That is, assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ when $k \geq 1$. We want to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$.

Inductive Step: Notice that

$$\begin{aligned}
\sum_{i=1}^{k} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + k + 1\right) \\
&= (k+1)\frac{2k^2 + k + 6(k+1)}{6} \\
&= (k+1)\frac{2k^2 + 7k + 6}{6} \\
&= (k+1)\frac{(k+2)(2k+3)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Thus $P(k+1)$ is true.

Summary: We showed that $P(1)$ is true and that whenever $k \geq 1$, $P(k) \to P(k+1)$, therefore $P(n)$ is true for all $n \geq 1$ by induction. □

8.16. Use induction to prove that for all $n \geq 1$,

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = 2 + (n-1)2^{n+1}$$

Proof. We use mathematical induction. When $n = 1$ we have $1 \cdot 2 = 2$ and $2 + (1-1)2^2 = 2 + 0 = 2$ so the statement is true when $n = 1$. Next we will assume that

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + k \cdot 2^k = 2 + (k-1)2^{k+1}$$

holds for some $k \geq 1$ and show that this implies that

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + k \cdot 2^k + (k+1) \cdot 2^{k+1} = 2 + k2^{k+2}.$$

Observe that

$$\begin{aligned}
1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + k\cdot 2^k &= 2 + (k-1)2^{k+1}\\
1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + k\cdot 2^k + (k+1)2^{k+1} &= 2 + (k-1)2^{k+1} + (k+1)2^{k+1}\\
&= 2 + (k-1+k+1)2^{k+1}\\
&= 2 + 2k\cdot 2^{k+1}\\
&= 2 + k2^{k+2}.
\end{aligned}$$

It follows by induction that $1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + n\cdot 2^n = 2 + (n-1)2^{n+1}$ for all $n \geq 1$. □

8.18. This proof is basically correct. I would suggest two changes:

(a) The basis step starts by stating that $1\cdot 1! = (1+1)! - 1$, which is the result we're trying to prove here. It would be much better to start by showing that $1\cdot 1! = 1$ and then showing that $(1+1)! - 1 = 1$ separately and then pointing out that these are equal.

(b) Given that we're trying to prove a statement about n, it would be better in the inductive step to use a different variable such as k. This prevents the inductive hypothesis from nearly matching the statement we are trying to prove. The key concept is that in the inductive hypothesis we are working with an existential quantification (*some* integer $k \geq 1$), while the statement we're ultimately trying to prove is a universal quantification (*all* integers $n \geq 1$).

8.19. If $k > 1$ then $k \neq 1$ so $P(k) \to P(k+1)$ cannot be used to show $P(1) \to P(2)$.

8.21. This choice made the algebra less messy.

8.25. One problem is with the statement "assume that *any* collection of n goats are all the same color." If we're going to use induction we have to say "assume we have a collection of n goats that are all the same color." This is very different than saying assuming that as long as we have n goats they'll all be the same color.

Another problem occurs when going from the base $n = 1$ case to the two-goat case. Suppose $n = 1$ and introduce a second goat. Now the first group contains one goat, numbered "1." The second group also contains one goat, numbered "2." Notice that goat 2 is not in both collections.

8.26. *Proof 1:* The statement we're trying to prove states that the number of binary palindromes of length $2n$ is 2^n for all $n \geq 0$. The base case here should be $n = 0$, not $k = 1$ as stated in the proof. The writer should have used n rather than k and, more importantly, started with 0 rather than 1. Another problem has to do with how the inductive step is carried out. Since we're working with palindromes, when constructing a string of length $2(k+1)$ from a string of length $2k$, we should either prepend 0 to the front and append at 0 to the end, or do the same with 1. Putting 00 or 11 at either the beginning or the end will not generally result with a palindrome.

Proof 2: This is awful, with only very first statement making any sense. The statement of the base case is fine, but the statement "Now assume it is true for all n." is not. To what does "it" refer? We're trying to prove a statement for all $n \geq 0$, how can we justify assuming "it" is true for all n? Also, in this context the statement "adding a bit to either end" should be avoided - the operation is concatenation, not addition. Similarly problematic is "...which multiplies the total number by 2^2 permutations". A permutation is a rearrangement of an ordered tuple, so we can count permutations, but we cannot multiply by them. It's also not at all clear where the number 2^2 comes from. Throughout this attempted proof the writer uses "it" and "they" but it is not at all clear to what they refer.

Proof 3: This attempt is better in that I believe the writer is reasoning correctly. They have not, however, explained their reasoning clearly and unambiguously. As in Proof 2, here again the word "it" inappropriately.

8.27. *Proof.* We begin by noting there is exactly one empty string. It has length 0 and is trivially a palindrome, so there is exactly one binary palindrome of length 0. Since $2^0 = 1$, we see that statement holds for $n = 0$.

We now assume that there are 2^k binary palindromes of length $2k$ for some $k \geq 0$. For each of these, we can prepend and append a leading and trailing bit to generate a new string of length $2(k+1)$. For the new string to be a palindrome the new leading and trailing bits must be the same. Since we have two choices, either 0, or 1, each original palindrome of length $2k$ can be used to generate two binary palindromes of length $2(k+1)$. Thus, there are $2 \cdot 2^k = 2^{k+1}$ binary palindromes of length $2(k+1)$. This completes the induction. □

8.2 Recursion

8.3 Solving Recurrence Relations

8.4 Analyzing Recursive Algorithms