Mathematical Induction Problems

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1. (Problem 10 in text) For any integer $n \geq 0$, it follows that $3|(5^{2n} - 1)$.

2. (Problem 20 in text) Prove that

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

for every $n \in \mathbb{N}$.

3. (Problem 26 in text) Concerning the Fibonacci sequence, prove that

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

4. For any integer $n \geq 2$, it follows that $2^{3n} - 1$ is not prime (prove using induction).

Hint: To show an integer is not prime you need to show that it is a multiple of two natural numbers, neither of which is 1. It turns out that in this problem not only is $2^{3n} - 1$ not prime for all $n \geq 2$, it is a multiple of a particular integer, say k. Check enough cases so that you figure out what k should be and rephrase the problem as "For any integer $n \geq 2$, it follows that $k|(2^{3n} - 1)$."

5. (**Bonus question**) We define the *Pell Sequence* by the initial values $p_1 = 1$ and $p_2 = 2$ along with the recurrence relation

$$p_n = 2p_{n-1} + p_{n-2}.$$

This means that

$$\begin{aligned} p_1 &= 1 \\ p_2 &= 2 \\ p_3 &= 2p_2 + p_1 = 5 \\ p_4 &= 2p_3 + p_2 = 12 \\ p_5 &= 2p_4 + p_3 = 29 \\ &\vdots \end{aligned}$$

Proposition. *Prove by induction on n that* $2p_n^2+(-1)^n = (p_{n+1}-p_n)^2$.

Hint: The recurrence can be rewritten as

$$p_n - p_{n-1} = p_{n-1} + p_{n-2} \tag{1}$$

and the relation you need to prove can be written as

$$(p_{n+1} - p_n)^2 - 2p_n^2 = (-1)^n. \tag{2}$$

Proceed as usual: Let S_n be the statement given by (2). Show S_1 and S_2 are true. Now assume that $S_1, S_2, ..., S_k$ have all been validated and show S_{k+1} is true where

$$S_{k+1} : (p_{k+2} - p_{k+1})^2 - 2p_{k+1}^2 = (-1)^{k+1}. \tag{3}$$

Replace $p_{k+2} - p_{k+1}$ in the left-hand side of (3) using (1) with $n = k+2$ and expand and simplify. Show that this yields

$$-p_{k+1}^2 + 2p_{k+1}p_k + p_k^2.$$

Factor out a (-1) and complete the square. From there you should be able to use S_k (write it out if you need to), which we have assumed to be true, to finish things off.