Proofs Involving Sets

MAT231

Transition to Higher Mathematics

Fall 2014

Outline

Exercise 3

Proposition

If $k \in \mathbb{Z}$, then $\{n \in \mathbb{Z} : n|k\} \subseteq \{n \in \mathbb{Z} : n|k^2\}$.

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Proof.

Suppose $k \in \mathbb{Z}$ and let $K = \{n \in \mathbb{Z} : n|k\}$ and $S = \{n \in \mathbb{Z} : n|k^2\}$. Let $x \in K$ so that $x|k$. We can write $k = ax$ for some $a \in \mathbb{Z}$. Then $k^2 = (ax)^2 = x(a^2x)$ so $x|k^2$. Thus, $x \in S$. Since *any* element x in K is also in S, we know that *every* element x in K is also in S, thus $K \subseteq S$. □

Exercise 7

Proposition

Suppose A, B, and C are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.

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Proposition

Suppose A, B, *and* C *are sets. If* $B \subseteq C$, *then* $A \times B \subseteq A \times C$.

Proof.

Let sets A, B, and C be given with $B \subseteq C$. Then

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

Let $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$. Since $B \subseteq C$, we know $y \in C$, so it must be that $(x, y) \in A \times C$. Thus $A \times B \subseteq A \times C$. $\square$

Exercise 13

Proposition

If A, B, and C are sets, then $A - (B \cup C) = (A - B) \cap (A - C)$.

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Proposition

If A, B, and C are sets, then $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof.

Suppose A, B and C are sets and let $x \in A - (B \cup C)$. Then

$$\begin{aligned}
x \in A - (B \cup C) &\equiv (x \in A) \wedge (x \notin (B \cup C)) \\
&\equiv (x \in A) \wedge (x \in \overline{(B \cup C)}) \\
&\equiv (x \in A) \wedge (x \in (\overline{B} \cap \overline{C})) \\
&\equiv (x \in A) \wedge (x \in \overline{B}) \wedge (x \in \overline{C}) \\
&\equiv (x \in A \wedge x \in \overline{B}) \wedge (x \in A \wedge x \in \overline{C}) \\
&\equiv (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\
&\equiv x \in (A - B) \wedge x \in (A - C) \\
&\equiv x \in (A - B) \cap (A - C)
\end{aligned}$$

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If A, B, and C are sets, then $A - (B \cup C) = (A - B) \cap (A - C)$.

Proof.

(Continued) This result shows that $A - (B \cup C) \subseteq (A - B) \cap (A - C)$. To show $(A - B) \cap (A - C) \subseteq A - (B \cup C)$ we start with $x \in (A - B) \cap (A - C)$.

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Proposition

If A, B, and C are sets, then $A - (B \cup C) = (A - B) \cap (A - C)$.

Exercise 13

Proposition

If A, B, and C are sets, then $A-(B\cup C)=(A-B)\cap(A-C)$.

Proof.

(Continued)

$$\begin{aligned}
x\in(A-B)\cap(A-C) &\equiv (x\in(A-B))\wedge(x\in(A-C))\\
&\equiv (x\in A\wedge x\notin B)\wedge(x\in A\wedge x\notin C)\\
&\equiv (x\in A\wedge x\in \overline{B})\wedge(x\in A\wedge x\in \overline{C})\\
&\equiv (x\in A)\wedge(x\in \overline{B})\wedge(x\in \overline{C})\\
&\equiv (x\in A)\wedge(x\in(\overline{B}\cap\overline{C}))\\
&\equiv (x\in A)\wedge(x\in \overline{(B\cup C)})\\
&\equiv (x\in A)\wedge(x\notin(B\cup C))\\
&\equiv x\in A-(B\cup C)
\end{aligned}$$

Exercise 13

Proof.

(Continued) Since each set is a subset of the other, we have established the equality of the two sets so $A - (B \cup C) = (A - B) \cap (A - C)$. □

Exercise 13

This same proposition can be proved with a single derivation.

Proof.

Suppose A, B and C are sets. Then

$$\begin{aligned}
A - (B \cup C) &= \{x : x \in A - (B \cup C)\} \\
&= \{x : (x \in A) \wedge (x \notin (B \cup C))\} \\
&= \{x : (x \in A) \wedge (x \in \overline{(B \cup C)})\} \\
&= \{x : (x \in A) \wedge (x \in (\overline{B} \cap \overline{C}))\} \\
&= \{x : (x \in A) \wedge (x \in \overline{B}) \wedge (x \in \overline{C})\} \\
&= \{x : (x \in A \wedge x \in \overline{B}) \wedge (x \in A \wedge x \in \overline{C})\} \\
&= \{x : (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)\} \\
&= \{x : x \in (A - B) \wedge x \in (A - C)\} \\
&= \{x : x \in (A - B) \cap (A - C)\} \\
&= (A - B) \cap (A - C).
\end{aligned}$$

Example A

Proposition

$\{p : p \text{ is a prime number}\} \cap \{k^2 - 1 : k \in \mathbb{N}\} = \{3\}.$

Example A

Proposition

$\{p : p \textit{ is a prime number}\} \cap \{k^2 - 1 : k \in \mathbb{N}\} = \{3\}$.

Proof.

Let

$$x \in \{p : p \text{ is a prime number}\} \cap \{k^2 - 1 : k \in \mathbb{N}\}$$

so that x is prime and $x = k^2 - 1 = (k-1)(k+1)$. This shows that x has two factors.

Every prime number has two positive factors 1 and itself, so either $(k-1) = 1$ or $(k+1) = 1$. Since these factors must be positive we know $(k+1)$ cannot be 1 because this would mean $k = 0$. Thus $(k-1) = 1$ and therefore $k = 2$.

Thus $x = (2-1)(2+1) = 1 \cdot 3 = 3$, which is the only element of $\{p : p \text{ is a prime number}\} \cap \{k^2 - 1 : k \in \mathbb{N}\}$. □

Example B

Prove this proposition using a proof by contradiction.

Proposition

$\{2k + 1 : k \in \mathbb{N}\} \cap \{4k : k \in \mathbb{N}\} = \varnothing.$

Example B

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Proposition

$\{2k+1 : k \in \mathbb{N}\} \cap \{4k : k \in \mathbb{N}\} = \varnothing$.

Proof.

Suppose $\{2k+1 : k \in \mathbb{N}\} \cap \{4k : k \in \mathbb{N}\} \neq \varnothing$. Then some element x exists for which $x \in \{2k+1 : k \in \mathbb{N}\} \cap \{4k : k \in \mathbb{N}\}$ so that

$$x \in \{2k+1 : k \in \mathbb{N}\} \quad \text{and} \quad x \in \{4k : k \in \mathbb{N}\}$$

Since $x \in \{2k+1 : k \in \mathbb{N}\}$ we know that x has the form $2k+1$ for a natural number k and so by definition x is odd. However, since $x \in \{4k : k \in \mathbb{N}\}$ we know that x has the form $4k = 2(2k)$ which, since k and hence $2k$ are natural numbers, means that x is even. Since x cannot be both even and odd we have a contradiction. Therefore $\{2k+1 : k \in \mathbb{N}\} \cap \{4k : k \in \mathbb{N}\}$ cannot contain any element x, so it must be empty. □